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THE PROBLEM OF FINDING OF EIGENVECTORS FOR 4P-SH-SAW PROPAGATION IN 6 *mm* MEDIA

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ABSTRACT

This theoretical report is pertinent to the mathematical problem of finding of all the possible eigenvectors for the four-potential shear-horizontal surface acoustic wave (4P-SH-SAW) propagation in suitable solids. In this case, the wave propagation is coupled with the four potentials, i.e. the electrical, magnetic, gravitational, and cogravitational ones. The taking into account these four potentials results in significant difficulties to find any eigenvector because the mathematical method is significantly complicated. To find all suitable eigenvectors is very important here because it will allow one in the future to theoretically disclose all suitable solutions of acoustic waves. This is applicable to the problem of finding of propagation velocities of the SH-SAWs, interfacial SH-waves, plate SH-waves, and more complicated cases. It is thought that all the effects (for instance, the gravitocogravitic, gravitoelectric, cogravitoelectric, gravitomagnetic, cogravitomagnetic effects) individually or collaboratively participating in the acoustic wave propagation can be vital for acoustic wave propagation that can be readily used for constitution of suitable technical devices. This fact must be first demonstrated theoretically for experimentalists and engineers working with the transmitting, detecting, and converting of the electromagnetic waves' signals. It is expected that the future communication technologies will also exploit gravitational waves for the new communication era based on some gravitational phenomena.

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INTRODUCTION

2016 was the jubilee year, namely the centenary celebration of the prediction of the existence of gravitational waves. This prediction was done by Albert Einstein (1916). Also, 2016 is the year when Einstein's prediction was experimentally confirmed by a team of more than thousand researchers (Abbott *et al*., 2016). They were working during several decades since 1970s for the purpose to detect the gravitational waves in space experiments called the LIGO (Laser Interferometer Gravitational-Wave Observatory). Using obtained data in the space experiments, they presented the first direct detection of gravitational waves and the first observation of a binary black hole merger. The black holes are famous invisible objects possessing very strong gravitational fields that are strong enough to capture even the electromagnetic waves propagating in a vacuum with the speed of light.

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André Füzfa (2016) has reported his experimental realization when the magnetic and gravitational forces can interact: the magnetic field can control the gravitational field. He has described one revolutionary approach for the creation of gravitational fields from well-controlled magnetic fields and observing how these magnetic fields can bend space-time. He has proposed a theoretical device based on superconducting electromagnets for creation of detectable gravitational fields. He has also evaluated the coupling between the magnetic and gravitational fields in an order of ~ 10^{-35}. This leads to the generation of extremely weak gravitational redshift and gravitational attraction. Also, the atomic interferometry has been considered for the determination of the extremely faint change in the gravitational potential produced by small masses on matter waves. So, the amplitude of the extra gravitational acceleration artificially generated by the magnetic field of a single-layered solenoid is extremely weak but lies just a few orders of magnitude below the precision of atomic interferometry in the measurement of differential acceleration of 10^{-15} g. His pioneer work can release many new applications concerning telecommunications with gravitational waves. The ability to produce, detect, and control gravitational fields can certainly be a major achievement in modern physics. Therefore, scientific interest in the problem of interactions between the gravitational and electromagnetic waves continuously increases.

The author of this report (Zakharenko, 2016) has developed the original theory concerning the four-potential shear-horizontal surface acoustic wave (4P-SH-SAW) propagation, i.e. the waves coupled with the electrical, magnetic, gravitational, and cogravitational potentials. This means that all five fields (elastic, electrical, magnetic, gravitational, and cogravitational) contribute to the wave motion. There are also interactions among and between each pair of the mechanical, electrical, magnetic, gravitational, and cogravitational subsystems in the common thermodynamic system. Theoretical work (Zakharenko, 2016) treats two extra subsystems (gravitational and cogravitational) to the thermodynamic system for piezoelectromagnetics (PEMs) in which mechanical, electrical, and magnetic subsystem contribute. The piezoelectromagnetics are a class of well-known magnetoelectric materials. The SH-SAW propagation coupled with the electrical and magnetic potentials in the 6 *mm* PEMs represented a great interest in the last decade. The transversely-isotropic 6 *mm* materials and suitable propagation directions (Gulyaev, 1998) for the acoustic waves are well known. There is single review (Zakharenko, 2013a) on the PEM-SH-SAWs and the disclosed peculiarities for the problem of the wave propagation are discussed in (Zakharenko, 2013b). There is also single book (Zakharenko, 2010) on some new PEM-SH-SAWs. The book was published in 2010 under the influence of the discoveries done by Melkumyan (2007). Some extra new PEM-SH-SAWs were recently discovered in (Zakharenko, 2013c) and (Zakharenko, 2015a) and some of the new waves were analytically studied in (Zakharenko, 2015b). It worth noting that Melkumyan (2007) has also discovered the new PEM-SH-SAW called the surface Bleustein-Gulyaev-Melkumyan wave in order to have an analogy with the surface Bleustein-Gulyaev wave. The later SH-SAW can propagate in pure piezoelectrics or pure piezomagnetics and was simultaneously discovered by Bleustein (1968) and Gulyaev (1969).

The wave propagation studied in (Zakharenko, 2016) is caused by the contribution of the following five fields: the elastic, electric, magnetic, gravitational, and cogravitational. The last two fields are respectively known as the gravitoelectric and gravitomagnetic ones in the theory of the gravitoelectromagnetism. Paper (Zakharenko, 2016) uses gravitational and cogravitational instead of gravitoelectric and gravitomagnetic because the last two words are naturally used for the corresponding exchange effects between the gravitational and electric (magnetic) subsystems, respectively. In this introductory part, it is possible to briefly review some studies on the cogravitational (gravitomagnetic) field because this field of five can be the most infamous for the reader.

Researchers specializing in general relativity, gravitational theories, and cosmology have formed the existence necessity of a magnetic-like gravitational field unknown in other domains of physics. Heaviside (1893) has first hypothesized the existence of the cogravitational (gravitomagnetic) field. This extra field predicted by general relativity was first formulated in Thirring (1918), Lense and Thirring (1918) and Thirring (1921). The translation of these papers was introduced by Mashhoon *et al*. (1984). Forward (1961) has first expressed the gravitational field equations (with the gravitomagnetic field called the prorotational field) in a vector form directly analogous and nearly identical to Maxwell's equations for electromagnetism. DeWitt (1966) has first identified the significance of gravitational effects in a superconductor and demonstrated that a magnetic-type gravitational field must result in the presence of fluxoid quantization. Ross (1983) has substantially expanded DeWitt's work.

In the early 1970s, Wallace has issued three patents (Wallace, 1971a; Wallace, 1971b; Wallace, 1974) for some unusual inventions relating to the gravitational field. He has also developed an experimental apparatus for generating and detecting a secondary gravitational field called the kinemassic field, i.e. the gravitomagnetic field. He has here described three different methods used for detection of the gravitomagnetic field: (1) change in the motion of a body on a pivot, (2) detection of a transverse voltage in a semiconductor crystal, and (3) a change in the specific heat of a crystal having spin-aligned nuclei. Also, he has shown an analogy between the un-paired angular momentum in some materials (elements and isotopes possessing an odd number of nucleons) and the un-paired magnetic moments of electrons in ferromagnetics. Wallace believed that a gravitational shield can be created: the gravitomagnetic field can create a secondary gravitoelectric field leading to exclusion of an existing primary background gravitoelectric field. However, these detected gravitational shielding effects are extremely small.

There is gravitational theoreticians' bible (Misner *et al*., 1973). This book presents gravitational field equations derived from general relativity in a form similar to Maxwell's equations along with many other theories. It is necessary to state that the Maxwell-like equations for gravitation are relatively simple and can have possible practical applications. Therefore, these equations must be perfectly described in any undergraduate physics textbook that is currently missing. Braginsky *et al*. (1977) have written down gravitational field equations (with the gravitomagnetic field called the magnetic-type gravity) derived from the general relativity theory in a form similar to Maxwell's equations. A variety of experiments are proposed and analyzed for detecting the gravitomagnetic field. His further collaborative paper (Braginsky *et al*., 1984) analyses an experiment for detecting the earth's gravitomagnetic field. It is possible that the authors of papers (Braginsky *et al*., 1977; Braginsky *et al*., 1984) are the first who have utilized the terms "gravitoelectric" and "gravitomagnetic".

Bedford and Krumm (1985) have also derived the necessary existence of the gravitomagnetic field from arguments based on special relativity. Krumm and Bedford (1987) have also derived the gravitational Poynting vector and used the terms "gravinetic" and "gravistatic" for the gravitational fields. One year later, Kolbenstvedt (1988) has exploited the terms "gravielectric" and "gravimagnetic" for these fields and predicted the gravitomagnetic field existence using special relativity and time dilation. In the following year, Mashhoon *et al*. (1989) have provided a summary analysis of Maxwell's equations for gravitation and an in-depth analysis of the Gravity Probe-B orbital gyroscope experiment for detecting the earth's gravitomagnetic field. Harris (1991) has also composed Maxwell's equations for gravitation from general relativity in the case of nonrelativistic velocities and relatively weak field strengths.

In the book published by Jefimenko (1992), the electromagnetic field equations based on retarded sources (charges, moving charges, and accelerating charges) were derived and similar arguments to the gravitational field equations were applied. He also presents Maxwell's equations for gravitation and an unusual mass configuration relevant to an effect of change in gravity. Ciufolini and Wheeler (1995) have developed the electromagnetic analog of the gravitational field equations and provided an in-depth analysis of experiments for detecting the gravitomagnetic field. So, researchers (Misner *et al*., 1973; Braginsky *et al*., 1977; Braginsky *et al*., 1984; Bedford, and Krumm, 1985; Krumm and Bedford, 1987; Kolbenstvedt, 1988; Mashhoon *et al*., 1989; Harris, 1991; Jefimenko, 1992; Ciufolini and Wheeler, 1995) have demonstrated the necessary existence of the gravitomagnetic field, using arguments based on general relativity, special relativity, and the cause and effect relationship resulting from noninstantaneous propagation of energy (retardation).

Li and Torr (1991) have also presented Maxwell's equations for gravitation in a form where the gravitomagnetic permeability of a superconductor is different from the permeability of a vacuum (free space). They have derived an interrelationship between the magnetic and gravitomagnetic fields in a superconductor and established that an electrical current also results in a mass current. It is also found that the magnetic flux in a superconductor is a function of the gravitomagnetic permeability, and vice versa, and shown that the magnetically created gravitomagnetic field in a superconductor can be ~ 10^{11} times larger than the internal magnetic field. One year later, Li and Torr (1992) have discussed the interrelationship between the magnetic and cogravitational (gravitomagnetic) fields in superconductors, in which some spin alignment of the lattice ions can cause the later field. They have also estimated the propagation velocity of a gravitational wave in a superconductor: it is two orders of magnitude slower than the vacuum velocity. This allowed them to perform an estimation of the value of relative gravitomagnetic permeability of a superconductor: it is ~ 10,000 times larger than that for a vacuum. In the following year, Torr and Li (1993) have continued their analysis of gravitational effects in superconductors and shown a striking similarity to Wallace's ideas that the coherent alignment of lattice ion spins can generate detectable gravitomagnetic and gravitoelectric fields. Li *et al*. (1997) have described an experiment showing that the effect of change in gravity was very small, if it existed at all. In 1999, Li has left the University of Alabama and found her company AC Gravity LLC that remained listed as an "existent" business in 2014.

Nordtvedt (1988) has reported an indirect detection of the gravitomagnetic field by astronomical observations of the precession rate of the binary pulsar PSR 1913+16. Ciufolini *et al*. (1997) have reported that the gravitomagnetic field resulting from the earth's rotation was experimentally detected and measured by laser tracking of the LAGEOS II satellite. Their results agreed with the Lense-Thirring derivation from the general relativity theory. Ciufolini *et al*. (1998) have also reported a test of general relativity and measurement of the Lense-Thirring effect with two Earth satellites.

With a bulk $YBa_2Cu_3O_{7-x}$ superconductor, Podkletnov and Nieminen (1992) have described a possibility of gravitational force shielding when a 2% reduction in weight can be achieved in a mass suspended over a levitated and rotating toroidal-shaped type II superconductor disk. With the Meissner effect, constant vertical and time varying horizontal magnetic fields were applied for rapid rotation of the disk. Podkletnov's "gravity shielding" experiment at Tampere was replicated by the NASA and may also be an example of the effect described in Wallace's patents of the early 1970s claiming that a rotating object containing unpaired nuclear spins can modify gravity. It is a pleasure that an explanation in terms of a gravitational analogue to the magnetic field of electromagnetism is used.

Using the brief review written above, the reader is already familiar with the cogravitational (gravitomagnetic) field and different studies on the field. Therefore, it is possible to return to the problem of the acoustic wave propagation coupled with the four potentials (electric, magnetic, gravitational, and cogravitational). This theoretical report represents a complement to the previously published work Zakharenko (2016). Namely, it touches the very important mathematical problem of finding of the apt eigenvectors. To find all the possible eigenvectors is crucial because the utilization of different eigenvectors in coupe with different electrical, magnetic, gravitational, and cogravitational boundary conditions can lead to different formulas for calculation of propagation velocities of the acoustic wave. For instance, the four-potential shear-horizontal surface acoustic waves (4P-SH-SAWs) are the simplest example for the case. These 4P-SH-SAWs represent different mechanisms of instability of the corresponding bulk acoustic wave, i.e. 4P-SH-BAW. These different instability mechanisms can be caused even by extremely small exchange effects, for instance, the magnetoelectric, gravitocogravitic, gravitoelectric, cogravitoelectric, gravitomagnetic, and cogravitomagnetic effects. These effects represent an exchange between two corresponding subsystems of four: electric, magnetic, gravitational, and cogravitational.

This is possible that extremely weak effects can contribute in wave existence in a major way because the very weak magnetoelectric effect can be vital for the some acoustic wave propagation in piezoelectromagnetics (Zakharenko, 2010) in which the mechanical, electric, and magnetic subsystems interact. Therefore, let's resolve this mathematical problem concerning the finding of all possible eigenvectors. This represents a quite complicated mathematical task that will be demonstrated in the following sections.

The theory and the problem of finding of eigenvalues and eigenvectors

For the problem of acoustic wave propagation in solids, it is first necessary to resolve the equations of motion. This means that all suitable eigenvalues and corresponding eigenvectors must be disclosed. To resolve the equations of motion is a complicated task for the common case, for which they can be resolved only numerically. In the common case, the coupled equations of motion are written down in a tensor form representing the following compact form of the well-known Green-Christoffel equation (Zakharenko, 2016): $\left(GL_{IJ} - \delta_{IJ}\rho V_{ph}\right)U_I^0 = 0$, where the indices I and J run from 1 to 7 and ρ is the mass density. The phase velocity defined by $V_{ph} = \omega/k$ is proportional to the angular frequency ω and inversely proportional to the wavenumber k in the propagation direction. GL_{IJ} stands for the components of the modified symmetric tensor (Zakharenko, 2016) and δ_{IJ} represents the Kronecker delta-function with the following conditions: $\delta_{IJ} = 1$ for $I = J < 4$, $\delta_{IJ} = 0$ for $I \neq J$, and $\delta_{44} = \delta_{55} = \delta_{66} = \delta_{77} = 0$. Also, parameters U_I^0 represent the components of the eigenvector $\left(U_1^0, U_2^0, U_3^0, U_4^0, U_5^0, U_6^0, U_7^0\right)$. This compact tensor form of the coupled equations of motion represents the common problem for determination of the eigenvalues and eigenvectors.

However there are particular cases depending on the material symmetry and propagation directions when analytical solutions can be obtained. For the acoustic wave propagation in the transversely isotropic (6 *mm*) materials, the suitable propagation directions (Gulyaev, 1998; Dieulesaint and Royer, 1980; Auld, 1990) exist in many directions perpendicular to the sixfold symmetry axis. For this case, both the coupled equations of motion and the boundary conditions' determinant split into two independent parts. This allows one to separately study these two parts. The first part is for the case of the propagation of the purely mechanical wave, for instance, the surface Rayleigh type waves (Dieulesaint and Royer, 1980; Auld, 1990; Zakharenko, 2005). These acoustic waves with the in-plane polarization are famous and this work has no interest in their study. The second part is relevant to the propagation of the shear-horizontal (SH) acoustic wave with the anti-plane polarization. This case represents a great interest because the SH-wave propagation is couple with the electrical, magnetic, gravitational, and cogravitational potentials. This case was originally studied by Zakharenko (2016). To further develop the study by Zakharenko (2016), it is necessary to demonstrate all possible eigenvectors because each of them can lead to unique solution or the propagation velocity of the SH-wave.

Therefore, let's start to resolve the coupled equations of motion written down in the following tensor form (Zakharenko, 2016) for the case of the SH-wave propagation in the transversely isotropic (6 *mm*) materials:

$$\begin{pmatrix} C\left[m-\left(V_{ph}/V_{t4}\right)^2\right] & em & hm & gm & fm \\ em & -\varepsilon m & -\alpha m & -\zeta m & -\xi m \\ hm & -\alpha m & -\mu m & -\beta m & -\lambda m \\ gm & -\zeta m & -\beta m & -\gamma m & -\vartheta m \\ fm & -\xi m & -\lambda m & -\vartheta m & -\eta m \end{pmatrix} \begin{pmatrix} U^0 \\ \varphi^0 \\ \psi^0 \\ \Phi^0 \\ \Psi^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (1)$$

where $\left(U^0, \varphi^0, \psi^0, \Phi^0, \Psi^0\right) = \left(U_2^0, U_4^0, U_5^0, U_6^0, U_7^0\right)$, $m = 1 + n_3^2$, and $V_{t4} = \sqrt{C/\rho}$.

Table 1 lists all the material parameters present in equations (1). The values of the material parameters must be unique for each solid continuum. There is no necessity in this study to write down the values of the material parameters for a particular material. However, it is possible to provide the well-known vacuum material parameters listed in Table 2.

Table 1. The material parameters' dimensions.

Material parameter	Symbol	Dimension
Mass density	ρ	kg/m^3
Elastic stiffness constant	C	N/m^2
Piezoelectric constant	e	C/m^2
Piezomagnetic coefficient	h	T = N/(A×m)
Piezogravitic constant	g	kg/m^2
Piezocogravitic coefficient	f	rad/s
Electric constant	ε	F/m
Magnetic constant	μ	N×s^2/C^2
Electromagnetic constant	α	N×s/(V×C)
Gravitic constant	γ	kg^2/(N×m^2)
Cogravitic constant	η	m/kg
Gravitocogravitic constant	ϑ	s/m
Gravitoelectric constant	ζ	C×kg/(J×m)
Cogravitoelectric constant	ξ	m/Wb
Gravitomagnetic constant	β	T×kg×m/J
Cogravitomagnetic constant	λ	T×m^3/(C×Wb)

Table 2. The vacuum parameters (Yavorsky *et al*., 2006), where the value of the vacuum elastic constant was borrowed from work by Kiang and Tong (2010).

Vacuum parameter	Value
Elastic constant	C_0 = 0.001 [N/m^2]
Electric constant (dielectric permittivity constant)	ε_0 = 0.08854187817 × 10^{-10} [F/m]
Magnetic constant (magnetic permeability constant)	μ_0 = 1.25663706144 × 10^{-6} [H/m]
Gravitic constant (gravitoelectric permittivity coefficient)	γ_0 = 1.498334 × 10^{10} [kg×s^2/m^3]
Cogravitic constant (gravitomagnetic permeability coefficient)	η_0 = 0.0742592 × 10^{-26} [m/kg]
Newtonian gravitational (gravitoelectric) constant	G_0 = 1/γ_0 = 0.667408 × 10^{-10} [m^3/(kg×s^2)]
Cogravitational (gravitomagnetic) constant	M_0 = 1/η_0 = 13.46635 × 10^{26} [kg/m]
Speed of light	$C_L = (G_0M_0)^{1/2} = (\gamma_0\eta_0)^{-1/2} = (\varepsilon_0\mu_0)^{-1/2}$ = 2.997924 × 10^8 [m/s]

This set (1) of five homogeneous equations allows one to determine all the eigenvalues n_3. With each of the found eigenvalues, it is possible to obtain the corresponding eigenvector $\left(U^0,\varphi^0,\psi^0,\Phi^0,\Psi^0\right)$. The suitable eigenvalues n_3 can be found when the determinant of the coefficient matrix in equations (1) is equal to zero. This determinant can be composed in the following convenient form consisting of five cofactors:

$$0 = m\times m\times m\times m \times \begin{vmatrix} C\left[m-\left(V_{ph}/V_{t4}\right)^2\right] & e & h & g & f \\ em & -\varepsilon & -\alpha & -\zeta & -\xi \\ hm & -\alpha & -\mu & -\beta & -\lambda \\ gm & -\zeta & -\beta & -\gamma & -\vartheta \\ fm & -\xi & -\lambda & -\vartheta & -\eta \end{vmatrix} \quad (2)$$

Therefore, these five factors in equation (2) give the following five pairs of the eigenvalues:

$$n_3^{(1,2)} = n_3^{(3,4)} = n_3^{(5,6)} = n_3^{(7,8)} = \mp \mathrm{j} \quad (3)$$

$$n_3^{(9,10)} = \mp \mathrm{j}\sqrt{1-\left(V_{ph}/V_{temgc}\right)^2} \quad (4)$$

In equations (3) and (4), j = (– 1)$^{1/2}$ is the imaginary unity. Also, expression (4) introduces the velocity (V_{temgc}) of the shear-horizontal bulk acoustic wave (SH-BAW) coupled with the electrical, magnetic, gravitational, and cogravitational potentials, i.e. the 4P-SH-BAW speed. It is defined by

$$V_{temgc} = \sqrt{C\left(1+K_{emgc}^2\right)/\rho} \quad (5)$$

In definition (5), K_{emgc}^2 defines the coefficient of the electromagnetogravitocogravitomechanical coupling (CEMGCMC). Its value can be calculated with the following formula:

$$K_{emgc}^2 = \frac{Z_1}{Z_2} \quad (6)$$

where

$$\begin{aligned} Z_1 &= e^2\left(\mu\gamma\eta + 2\beta\lambda\vartheta - \lambda^2\gamma - \beta^2\eta - \vartheta^2\mu\right) \\ &+ h^2\left(\varepsilon\gamma\eta + 2\zeta\xi\vartheta - \vartheta^2\varepsilon - \zeta^2\eta - \xi^2\gamma\right) \\ &+ g^2\left(\varepsilon\mu\eta + 2\alpha\xi\lambda - \lambda^2\varepsilon - \alpha^2\eta - \xi^2\mu\right) \\ &+ f^2\left(\varepsilon\mu\gamma + 2\alpha\beta\zeta - \beta^2\varepsilon - \alpha^2\gamma - \zeta^2\mu\right) \\ &+ 2eh\left(\vartheta^2\alpha + \zeta\beta\eta + \xi\gamma\lambda - \alpha\gamma\eta - \zeta\lambda\vartheta - \xi\beta\vartheta\right) \\ &+ 2eg\left(\alpha\beta\eta + \lambda^2\zeta + \xi\vartheta\mu - \alpha\lambda\vartheta - \zeta\mu\eta - \xi\beta\lambda\right) \\ &+ 2ef\left(\alpha\gamma\lambda + \zeta\vartheta\mu + \beta^2\xi - \alpha\beta\vartheta - \zeta\beta\lambda - \xi\mu\gamma\right) \\ &+ 2hg\left(\varepsilon\lambda\vartheta + \zeta\alpha\eta + \xi^2\beta - \varepsilon\eta\beta - \zeta\lambda\xi - \xi\vartheta\alpha\right) \\ &+ 2hf\left(\varepsilon\beta\vartheta + \zeta^2\lambda + \xi\alpha\gamma - \varepsilon\lambda\gamma - \zeta\vartheta\alpha - \xi\zeta\beta\right) \\ &+ 2gf\left(\varepsilon\beta\lambda + \alpha^2\vartheta + \xi\mu\zeta - \varepsilon\mu\vartheta - \alpha\zeta\lambda - \alpha\beta\xi\right) \end{aligned} \quad (7)$$

$$Z_2 = C(\varepsilon\mu - \alpha^2)(\gamma\eta - \vartheta^2) + C(\beta^2\xi^2 - \xi^2\mu\gamma - \beta^2\varepsilon\eta) + C(\lambda^2\zeta^2 - \lambda^2\varepsilon\gamma - \zeta^2\mu\eta) + 2C(\gamma\alpha\xi\lambda + \eta\alpha\beta\zeta + \varepsilon\beta\lambda\vartheta + \mu\zeta\xi\vartheta - \zeta\xi\beta\lambda - \alpha\zeta\lambda\vartheta - \alpha\beta\xi\vartheta) \quad (8)$$

Thus, all the possible eigenvalues, namely the five pairs defined by expressions (3) and (4) are already obtained. Each found eigenvalue n_3 must be now used in equation (1) anew to determine the corresponding eigenvector $(U^0, \varphi^0, \psi^0, \Phi^0, \Psi^0)$. It is obvious that each of two eigenvalues in each pair of five provides the same eigenvector because equations (1) depend on n_3^2. Also, four pairs (3) are identical and therefore, they must give the same set of the eigenvector components. Fifth pair (4) can certainly provide a unique set of the eigenvector components different from those for eigenvalues (3). So, only two eigenvectors must be found: the first will correspond to each of eight eigenvalues (3) and the second will correspond to the last pair of eigenvalues (4). Probably, this peculiarity can significantly simplify the problem. The second peculiarity is the situation that any of the eigenvector components does not depend on the phase velocity V_{ph}. The reader can check this statement by using the final expressions for the eigenvector components obtained in the following six sections. It is worth noting that this second peculiarity exists only for the transversely isotropic (6 *mm*) materials and results in many possible solutions for the propagating velocity. A great interest represents to find some solutions with a dramatic dependence on (one of) the following extremely weak exchange effects: the gravitocogravitic, gravitoelectric, cogravitoelectric, gravitomagnetic, and cogravitomagnetic effects.

With equations (1), it is natural first to obtain common forms of the eigenvector components. Utilizing eigenvalue (3) or (4), these common forms will then give certain eigenvector components $(U^0, \varphi^0, \psi^0, \Phi^0, \Psi^0)$. It is natural to utilize the first equation in set (1) for determination of the eigenvector component U^0 as a function of the rest components φ^0, ψ^0, Φ^0, and Ψ^0. Consequently, this function reads:

$$U^0 = -m(e\varphi^0 + h\psi^0 + g\Phi^0 + f\Psi^0)/CA \quad (9)$$

In definition (9), $m = 1 + n_3^2$ and the form of the parameter A depends on the form of the eigenvalue. For eigenvalues (3) and (4), the parameter A can respectively take the following forms:

$$A = m - (V_{ph}/V_{t4})^2 = -(V_{ph}/V_{t4})^2 \quad (10)$$

$$A = m - (V_{ph}/V_{t4})^2 = -mK_{emgc}^2 \quad (11)$$

Exploitation of definition (9) for equations' set (1) allows exclusion of the eigenvector component U^0 from the further consideration and to deal with a reduced set of four equations. This is the usual mathematical procedure for finding of the unknowns for the set of five equations in five unknowns. Let's treat the six different cases that lead to different sets of the eigenvector components. It is convenient to use definitions (10) and (11) of the parameter A only in the final expressions for the eigenvector components in the common forms.

The first case

In this case, the new reduced set of four homogeneous equations can be written as follows:

$$\varepsilon(1 + mK_e^2/A)\varphi^0 + \alpha(1 + mK_\alpha^2/A)\psi^0 + \zeta(1 + mK_\zeta^2/A)\Phi^0 + \xi(1 + mK_\xi^2/A)\Psi^0 = 0 \quad (12)$$

$$\alpha(1 + mK_\alpha^2/A)\varphi^0 + \mu(1 + mK_m^2/A)\psi^0 + \beta(1 + mK_\beta^2/A)\Phi^0 + \lambda(1 + mK_\lambda^2/A)\Psi^0 = 0 \quad (13)$$

$$\zeta(1 + mK_\zeta^2/A)\varphi^0 + \beta(1 + mK_\beta^2/A)\psi^0 + \gamma(1 + mK_g^2/A)\Phi^0 + \vartheta(1 + mK_\vartheta^2/A)\Psi^0 = 0 \quad (14)$$

$$\xi(1 + mK_\xi^2/A)\varphi^0 + \lambda(1 + mK_\lambda^2/A)\psi^0 + \vartheta(1 + mK_\vartheta^2/A)\Phi^0 + \eta(1 + mK_f^2/A)\Psi^0 = 0 \quad (15)$$

where

$$K_e^2 = e^2/C\varepsilon \quad (16)$$

$$K_m^2 = h^2/C\mu \quad (17)$$

$$K_\alpha^2 = eh/C\alpha \quad (18)$$

$$K_g^2 = g^2/C\gamma \quad (19)$$

$$K_f^2 = f^2/C\eta \quad (20)$$

$$K_\vartheta^2 = gf/C\vartheta \quad (21)$$

$$K_\zeta^2 = eg/C\zeta \quad (22)$$

$$K_\xi^2 = ef/C\xi \quad (23)$$

$$K_\lambda^2 = hf/C\lambda \quad (24)$$

$$K_\beta^2 = hg/C\beta \quad (25)$$

Next, treating equation (12) it is possible to determine the second eigenvector component φ^0 as a function of the components ψ^0, Φ^0, and Ψ^0. It can be composed as follows:

$$\varphi^0 = -\frac{\alpha(A + mK_\alpha^2)}{\varepsilon(A + mK_e^2)}\psi^0 - \frac{\zeta(A + mK_\zeta^2)}{\varepsilon(A + mK_e^2)}\Phi^0 - \frac{\xi(A + mK_\xi^2)}{\varepsilon(A + mK_e^2)}\Psi^0 \quad (26)$$

Definition (26) for φ^0 can be then utilized in equations (13), (14), and (15) to reduce the set of four homogeneous equations in four undetermined. As a result, the new reduced set of three homogeneous equations with three unknown components ψ^0, Φ^0, and Ψ^0 can be composed as follows:

$$\begin{gathered}\left(\frac{\mu\left(A+mK_m^2\right)}{A}-\frac{\alpha^2\left(A+mK_\alpha^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}\right)\psi^0\\+\left(\frac{\beta\left(A+mK_\beta^2\right)}{A}-\frac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\Phi^0\\+\left(\frac{\lambda\left(A+mK_\lambda^2\right)}{A}-\frac{\alpha\xi\left(A+mK_\alpha^2\right)\left(A+mK_\xi^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\Psi^0=0\end{gathered} \tag{27}$$

$$\begin{gathered}\left(\frac{\beta\left(A+mK_\beta^2\right)}{A}-\frac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\psi^0\\+\left(\frac{\gamma\left(A+mK_g^2\right)}{A}-\frac{\zeta^2\left(A+mK_\zeta^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}\right)\Phi^0\\+\left(\frac{\vartheta\left(A+mK_\vartheta^2\right)}{A}-\frac{\xi\zeta\left(A+mK_\xi^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\Psi^0=0\end{gathered} \tag{28}$$

$$\begin{gathered}\left(\frac{\lambda\left(A+mK_\lambda^2\right)}{A}-\frac{\alpha\xi\left(A+mK_\alpha^2\right)\left(A+mK_\xi^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\psi^0\\+\left(\frac{\vartheta\left(A+mK_\vartheta^2\right)}{A}-\frac{\xi\zeta\left(A+mK_\xi^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\Phi^0\\+\left(\frac{\eta\left(A+mK_f^2\right)}{A}-\frac{\xi^2\left(A+mK_\xi^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}\right)\Psi^0=0\end{gathered} \tag{29}$$

Exploiting equation (27), the third eigenvector component ψ^0 represents the following function of the eigenvector components Φ^0 and Ψ^0:

$$\begin{gathered}\psi^0=-\frac{\beta\left(A+mK_\beta^2\right)-\dfrac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{\varepsilon\left(A+mK_e^2\right)}}{\mu\left(A+mK_m^2\right)-\dfrac{\alpha^2\left(A+mK_\alpha^2\right)^2}{\varepsilon\left(A+mK_e^2\right)}}\Phi^0\\-\frac{\lambda\left(A+mK_\lambda^2\right)-\dfrac{\alpha\xi\left(A+mK_\alpha^2\right)\left(A+mK_\xi^2\right)}{\varepsilon\left(A+mK_e^2\right)}}{\mu\left(A+mK_m^2\right)-\dfrac{\alpha^2\left(A+mK_\alpha^2\right)^2}{\varepsilon\left(A+mK_e^2\right)}}\Psi^0\end{gathered} \tag{30}$$

Finally, definition (30) must be used for substitution in equations (28) and (29). This substitution results in the final two homogeneous equations in two unknowns: Φ^0 and Ψ^0. With these two equations, both Φ^0 and Ψ^0 can be readily defined. These two complicated equations can be composed in the following forms:

$$\begin{gathered}0=\left(\frac{\gamma\left(A+mK_g^2\right)}{A}-\frac{\zeta^2\left(A+mK_\zeta^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}-\frac{\left(\dfrac{\beta\left(A+mK_\beta^2\right)}{A}-\dfrac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)^2}{\dfrac{\mu\left(A+mK_m^2\right)}{A}-\dfrac{\alpha^2\left(A+mK_\alpha^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}}\right)\Phi^0\\+\left(\frac{\vartheta\left(A+mK_\vartheta^2\right)}{A}-\frac{\xi\zeta\left(A+mK_\xi^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\Psi^0\\-\frac{\left(\dfrac{\beta\left(A+mK_\beta^2\right)}{A}-\dfrac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\left(\lambda\left(A+mK_\lambda^2\right)-\dfrac{\alpha\xi\left(A+mK_\alpha^2\right)\left(A+mK_\xi^2\right)}{\varepsilon\left(A+mK_e^2\right)}\right)}{\mu\left(A+mK_m^2\right)-\dfrac{\alpha^2\left(A+mK_\alpha^2\right)^2}{\varepsilon\left(A+mK_e^2\right)}}\Psi^0\end{gathered} \tag{31}$$

$$\begin{gathered}0=\left(\frac{\vartheta\left(A+mK_\vartheta^2\right)}{A}-\frac{\xi\zeta\left(A+mK_\xi^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\Phi^0\\-\frac{\left(\dfrac{\beta\left(A+mK_\beta^2\right)}{A}-\dfrac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\left(\lambda\left(A+mK_\lambda^2\right)-\dfrac{\alpha\xi\left(A+mK_\alpha^2\right)\left(A+mK_\xi^2\right)}{\varepsilon\left(A+mK_e^2\right)}\right)}{\mu\left(A+mK_m^2\right)-\dfrac{\alpha^2\left(A+mK_\alpha^2\right)^2}{\varepsilon\left(A+mK_e^2\right)}}\Phi^0\\+\left(\frac{\eta\left(A+mK_f^2\right)}{A}-\frac{\xi^2\left(A+mK_\xi^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}-\frac{\left(\dfrac{\lambda\left(A+mK_\lambda^2\right)}{A}-\dfrac{\alpha\xi\left(A+mK_\alpha^2\right)\left(A+mK_\xi^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)^2}{\dfrac{\mu\left(A+mK_m^2\right)}{A}-\dfrac{\alpha^2\left(A+mK_\alpha^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}}\right)\Psi^0\end{gathered} \tag{32}$$

Equations (31) and (32) represent a set of two homogeneous equations in two unknowns: Φ^0 and Ψ^0. This pair of equations can be schematically written as follows: $a_1x + by = 0$ and $bx + a_2y = 0$. Therefore, the unknowns x and y can be chosen in two different ways:

(1) $x = -b$ and $y = a_1$;
(2) $x = a_2$ and $y = -b$.

Taking into account this fact it is natural to write down below two different sets of the eigenvector components for this case. With equation (31) and definitions (9), (26), and (30), the first eigenvectors can be composed. For eigenvalues (3), $m = 0$ and therefore, the corresponding eigenvector components are relatively simple, i.e.

$$\begin{pmatrix}U^{0(1)}\\ \varphi^{0(1)}\\ \psi^{0(1)}\\ \Phi^{0(1)}\\ \Psi^{0(1)}\end{pmatrix}=\begin{pmatrix}U^{0(3)}\\ \varphi^{0(3)}\\ \psi^{0(3)}\\ \Phi^{0(3)}\\ \Psi^{0(3)}\end{pmatrix}=\begin{pmatrix}U^{0(5)}\\ \varphi^{0(5)}\\ \psi^{0(5)}\\ \Phi^{0(5)}\\ \Psi^{0(5)}\end{pmatrix}=\begin{pmatrix}U^{0(7)}\\ \varphi^{0(7)}\\ \psi^{0(7)}\\ \Phi^{0(7)}\\ \Psi^{0(7)}\end{pmatrix}=\begin{pmatrix}U^0=0\\ \varphi^0=-\dfrac{\alpha}{\varepsilon}\psi^0-\dfrac{\zeta}{\varepsilon}\Phi^0-\dfrac{\xi}{\varepsilon}\Psi^0\\ \psi^0=-\dfrac{\beta-\dfrac{\alpha\zeta}{\varepsilon}}{\mu-\dfrac{\alpha^2}{\varepsilon}}\Phi^0-\dfrac{\lambda-\dfrac{\alpha\xi}{\varepsilon}}{\mu-\dfrac{\alpha^2}{\varepsilon}}\Psi^0\\ =-\dfrac{\left(\beta-\dfrac{\alpha\zeta}{\varepsilon}\right)\left(\vartheta-\dfrac{\xi\zeta}{\varepsilon}\right)-\left(\lambda-\dfrac{\alpha\xi}{\varepsilon}\right)\left(\gamma-\dfrac{\zeta^2}{\varepsilon}\right)}{\mu-\dfrac{\alpha^2}{\varepsilon}}\\ \Phi^0=\vartheta-\dfrac{\xi\zeta}{\varepsilon}-\dfrac{\left(\beta-\dfrac{\alpha\zeta}{\varepsilon}\right)\left(\lambda-\dfrac{\alpha\xi}{\varepsilon}\right)}{\mu-\dfrac{\alpha^2}{\varepsilon}}\\ \Psi^0=-\gamma+\dfrac{\zeta^2}{\varepsilon}+\dfrac{\left(\beta-\dfrac{\alpha\zeta}{\varepsilon}\right)^2}{\mu-\dfrac{\alpha^2}{\varepsilon}}\end{pmatrix} \tag{33}$$

However, for eigenvalue (4) there is a more complicated eigenvector. For this case, the utilization of definition (11), equation (31), and definitions (9), (26), (30) leads to the following complicated eigenvector components:

$$\begin{pmatrix} U^{0(9)} \\ \varphi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \left(\begin{array}{c} U^0 = \left(e\varphi^0 + h\psi^0 + g\Phi^0 + f\Psi^0\right) / \left(CK_{emgc}^2\right) \\ \varphi^0 = -\frac{\alpha K_A}{\varepsilon K_E}\psi^0 - \frac{\zeta K_Z}{\varepsilon K_E}\Phi^0 - \frac{\xi K_S}{\varepsilon K_E}\Psi^0 \\ \psi^0 = -\frac{\beta K_B - \frac{\alpha\zeta K_A K_Z}{\varepsilon K_E}}{\mu K_M - \frac{\alpha^2 K_A{}^2}{\varepsilon K_E}}\Phi^0 - \frac{\lambda K_L - \frac{\alpha\xi K_A K_S}{\varepsilon K_E}}{\mu K_M - \frac{\alpha^2 K_A{}^2}{\varepsilon K_E}}\Psi^0 \\ = -\frac{\left(\beta K_B - \frac{\alpha\zeta K_A K_Z}{\varepsilon K_E}\right)\left(\frac{\vartheta K_T}{K_{emgc}^2} - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E K_{emgc}^2}\right) - \left(\lambda K_L - \frac{\alpha\xi K_A K_S}{\varepsilon K_E}\right)\left(\frac{\gamma K_G}{K_{emgc}^2} - \frac{\zeta^2 K_Z{}^2}{\varepsilon K_E K_{emgc}^2}\right)}{\mu K_M - \frac{\alpha^2 K_A{}^2}{\varepsilon K_E}} \\ \Phi^0 = \frac{\vartheta K_T}{K_{emgc}^2} - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E K_{emgc}^2} - \frac{\left(\frac{\beta K_B}{K_{emgc}^2} - \frac{\alpha\zeta K_A K_Z}{\varepsilon K_E K_{emgc}^2}\right)\left(\lambda K_L - \frac{\alpha\xi K_A K_S}{\varepsilon K_E}\right)}{\mu K_M - \frac{\alpha^2 K_A{}^2}{\varepsilon K_E}} \\ \Psi^0 = -\frac{\gamma K_G}{K_{emgc}^2} + \frac{\zeta^2 K_Z{}^2}{\varepsilon K_E K_{emgc}^2} + \frac{\left(\frac{\beta K_B}{K_{emgc}^2} - \frac{\alpha\zeta K_A K_Z}{\varepsilon K_E K_{emgc}^2}\right)^2}{\frac{\mu K_M}{K_{emgc}^2} - \frac{\alpha^2 K_A{}^2}{\varepsilon K_E K_{emgc}^2}} \end{array} \right) \tag{34}$$

where

$$K_M = K_{emgc}^2 - K_m^2 \tag{35}$$

$$K_E = K_{emgc}^2 - K_e^2 \tag{36}$$

$$K_F = K_{emgc}^2 - K_f^2 \tag{37}$$

$$K_G = K_{emgc}^2 - K_g^2 \tag{38}$$

$$K_T = K_{emgc}^2 - K_\vartheta^2 \tag{39}$$

$$K_A = K_{emgc}^2 - K_\alpha^2 \tag{40}$$

$$K_S = K_{emgc}^2 - K_\xi^2 \tag{41}$$

$$K_Z = K_{emgc}^2 - K_\zeta^2 \tag{42}$$

$$K_B = K_{emgc}^2 - K_\beta^2 \tag{43}$$

$$K_L = K_{emgc}^2 - K_\lambda^2 \tag{44}$$

To obtain the second eigenvectors, it is necessary to use equation (32). Therefore, two eigenvectors corresponding to eigenvalues (3) and (4) can be respectively inscribed as follows:

$$\begin{pmatrix} U^{0(1)} \\ \varphi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \varphi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \varphi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(7)} \\ \varphi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix}$$

$$= \left(\begin{array}{c} U^0 = 0 \\ \varphi^0 = -\frac{\alpha}{\varepsilon}\psi^0 - \frac{\zeta}{\varepsilon}\Phi^0 - \frac{\xi}{\varepsilon}\Psi^0 \\ \psi^0 = -\frac{\beta - \frac{\alpha\zeta}{\varepsilon}}{\mu - \frac{\alpha^2}{\varepsilon}}\Phi^0 - \frac{\lambda - \frac{\alpha\xi}{\varepsilon}}{\mu - \frac{\alpha^2}{\varepsilon}}\Psi^0 \\ = \frac{\left(\beta - \frac{\alpha\zeta}{\varepsilon}\right)\left(\eta - \frac{\xi^2}{\varepsilon}\right) - \left(\lambda - \frac{\alpha\xi}{\varepsilon}\right)\left(\vartheta - \frac{\xi\zeta}{\varepsilon}\right)}{\mu - \frac{\alpha^2}{\varepsilon}} \\ \Phi^0 = -\eta + \frac{\xi^2}{\varepsilon} + \frac{\left(\lambda - \frac{\alpha\xi}{\varepsilon}\right)^2}{\mu - \frac{\alpha^2}{\varepsilon}} \\ \Psi^0 = \vartheta - \frac{\xi\zeta}{\varepsilon} - \frac{\left(\beta - \frac{\alpha\zeta}{\varepsilon}\right)\left(\lambda - \frac{\alpha\xi}{\varepsilon}\right)}{\mu - \frac{\alpha^2}{\varepsilon}} \end{array} \right) \tag{45}$$

$$\begin{pmatrix} U^{0(9)} \\ \varphi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \left(\begin{array}{c} U^0 = \left(e\varphi^0 + h\psi^0 + g\Phi^0 + f\Psi^0\right) / \left(CK_{emgc}^2\right) \\ \varphi^0 = -\frac{\alpha K_A}{\varepsilon K_E}\psi^0 - \frac{\zeta K_Z}{\varepsilon K_E}\Phi^0 - \frac{\xi K_S}{\varepsilon K_E}\Psi^0 \\ \psi^0 = -\frac{\beta K_B - \frac{\alpha\zeta K_A K_Z}{\varepsilon K_E}}{\mu K_M - \frac{\alpha^2 K_A{}^2}{\varepsilon K_E}}\Phi^0 - \frac{\lambda K_L - \frac{\alpha\xi K_A K_S}{\varepsilon K_E}}{\mu K_M - \frac{\alpha^2 K_A{}^2}{\varepsilon K_E}}\Psi^0 \\ = \frac{\left(\beta K_B - \frac{\alpha\zeta K_A K_Z}{\varepsilon K_E}\right)\left(\frac{\eta K_F}{K_{emgc}^2} - \frac{\xi^2 K_S{}^2}{\varepsilon K_E K_{emgc}^2}\right) - \left(\lambda K_L - \frac{\alpha\xi K_A K_S}{\varepsilon K_E}\right)\left(\frac{\vartheta K_T}{K_{emgc}^2} - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E K_{emgc}^2}\right)}{\mu K_M - \frac{\alpha^2 K_A{}^2}{\varepsilon K_E}} \\ \Phi^0 = -\frac{\eta K_F}{K_{emgc}^2} + \frac{\xi^2 K_S{}^2}{\varepsilon K_E K_{emgc}^2} + \frac{\left(\frac{\lambda K_L}{K_{emgc}^2} - \frac{\alpha\xi K_A K_S}{\varepsilon K_E K_{emgc}^2}\right)^2}{\frac{\mu K_M}{K_{emgc}^2} - \frac{\alpha^2 K_A{}^2}{\varepsilon K_E K_{emgc}^2}} \\ \Psi^0 = \frac{\vartheta K_T}{K_{emgc}^2} - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E K_{emgc}^2} - \frac{\left(\frac{\beta K_B}{K_{emgc}^2} - \frac{\alpha\zeta K_A K_Z}{\varepsilon K_E K_{emgc}^2}\right)\left(\lambda K_L - \frac{\alpha\xi K_A K_S}{\varepsilon K_E}\right)}{\mu K_M - \frac{\alpha^2 K_A{}^2}{\varepsilon K_E}} \end{array} \right) \tag{46}$$

One can find that obtained eigenvectors (33), (34), (45), and (46) depend only on the material parameters and do not depend on the phase velocity V_{ph}. All the material parameters are listed in table 1.

The second case

It is also possible to regroup equations (12), (13), (14), (15) and then to find new sets of the eigenvector components. So, the new regrouped set of four homogeneous equations can be written as follows:

$$
\begin{cases}
\varepsilon\left(1+mK_e^2/A\right)\varphi^0+\zeta\left(1+mK_\zeta^2/A\right)\Phi^0 \\
+\alpha\left(1+mK_\alpha^2/A\right)\psi^0+\xi\left(1+mK_\xi^2/A\right)\Psi^0=0 \\
\zeta\left(1+mK_\zeta^2/A\right)\varphi^0+\gamma\left(1+mK_g^2/A\right)\Phi^0 \\
+\beta\left(1+mK_\beta^2/A\right)\psi^0+\vartheta\left(1+mK_\vartheta^2/A\right)\Psi^0=0 \\
\alpha\left(1+mK_\alpha^2/A\right)\varphi^0+\beta\left(1+mK_\beta^2/A\right)\Phi^0 \\
+\mu\left(1+mK_m^2/A\right)\psi^0+\lambda\left(1+mK_\lambda^2/A\right)\Psi^0=0 \\
\xi\left(1+mK_\xi^2/A\right)\varphi^0+\vartheta\left(1+mK_\vartheta^2/A\right)\Phi^0 \\
+\lambda\left(1+mK_\lambda^2/A\right)\psi^0+\eta\left(1+mK_f^2/A\right)\Psi^0=0
\end{cases}
\tag{47}
$$

Similarly, from the first equation in set (47) it can be written the following dependence:

$$
\varphi^0=-\frac{\zeta\left(A+mK_\zeta^2\right)}{\varepsilon\left(A+mK_e^2\right)}\Phi^0
-\frac{\alpha\left(A+mK_\alpha^2\right)}{\varepsilon\left(A+mK_e^2\right)}\psi^0-\frac{\xi\left(A+mK_\xi^2\right)}{\varepsilon\left(A+mK_e^2\right)}\Psi^0
\tag{48}
$$

Using definition (48) for the second, third, and fourth equations in set (47), one can get the following reduced set of three homogeneous equations:

$$
\begin{cases}
\left(\frac{\gamma\left(A+mK_g^2\right)}{A}-\frac{\zeta^2\left(A+mK_\zeta^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}\right)\Phi^0+\left(\frac{\beta\left(A+mK_\beta^2\right)}{A}-\frac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\psi^0 \\
+\left(\frac{\vartheta\left(A+mK_\vartheta^2\right)}{A}-\frac{\xi\zeta\left(A+mK_\xi^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\Psi^0=0 \\
\left(\frac{\beta\left(A+mK_\beta^2\right)}{A}-\frac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\Phi^0+\left(\frac{\mu\left(A+mK_m^2\right)}{A}-\frac{\alpha^2\left(A+mK_\alpha^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}\right)\psi^0 \\
+\left(\frac{\lambda\left(A+mK_\lambda^2\right)}{A}-\frac{\alpha\xi\left(A+mK_\alpha^2\right)\left(A+mK_\xi^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\Psi^0=0 \\
\left(\frac{\vartheta\left(A+mK_\vartheta^2\right)}{A}-\frac{\xi\zeta\left(A+mK_\xi^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\Phi^0+\left(\frac{\lambda\left(A+mK_\lambda^2\right)}{A}-\frac{\alpha\xi\left(A+mK_\alpha^2\right)\left(A+mK_\xi^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\psi^0 \\
+\left(\frac{\eta\left(A+mK_f^2\right)}{A}-\frac{\xi^2\left(A+mK_\xi^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}\right)\Psi^0=0
\end{cases}
\tag{49}
$$

It is natural to exploit the first equation in set (49) for the determination of the eigenvector component Φ^0. It is defined by

$$
\Phi^0=-\frac{\beta\left(A+mK_\beta^2\right)-\frac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{\varepsilon\left(A+mK_e^2\right)}}{\gamma\left(A+mK_g^2\right)-\frac{\zeta^2\left(A+mK_\zeta^2\right)^2}{\varepsilon\left(A+mK_e^2\right)}}\psi^0
-\frac{\vartheta\left(A+mK_\vartheta^2\right)-\frac{\xi\zeta\left(A+mK_\xi^2\right)\left(A+mK_\zeta^2\right)}{\varepsilon\left(A+mK_e^2\right)}}{\gamma\left(A+mK_g^2\right)-\frac{\zeta^2\left(A+mK_\zeta^2\right)^2}{\varepsilon\left(A+mK_e^2\right)}}\Psi^0
\tag{50}
$$

A substitution of the eigenvector component Φ^0 defined by (50) in the second and third equations in set (49) leads to the following final two homogeneous equation, with which it is already possible to soundly determine the rest two eigenvector components ψ^0 and Ψ^0:

$$
\begin{cases}
0=\left(\frac{\mu\left(A+mK_m^2\right)}{A}-\frac{\alpha^2\left(A+mK_\alpha^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}-\frac{\left(\frac{\beta\left(A+mK_\beta^2\right)}{A}-\frac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)^2}{\frac{\gamma\left(A+mK_g^2\right)}{A}-\frac{\zeta^2\left(A+mK_\zeta^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}}\right)\psi^0 \\
+\left(\frac{\lambda\left(A+mK_\lambda^2\right)}{A}-\frac{\alpha\xi\left(A+mK_\alpha^2\right)\left(A+mK_\xi^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\Psi^0 \\
-\left(\frac{\beta\left(A+mK_\beta^2\right)}{A}-\frac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\frac{\vartheta\left(A+mK_\vartheta^2\right)-\frac{\xi\zeta\left(A+mK_\xi^2\right)\left(A+mK_\zeta^2\right)}{\varepsilon\left(A+mK_e^2\right)}}{\gamma\left(A+mK_g^2\right)-\frac{\zeta^2\left(A+mK_\zeta^2\right)^2}{\varepsilon\left(A+mK_e^2\right)}}\Psi^0 \\
0=\left(\frac{\lambda\left(A+mK_\lambda^2\right)}{A}-\frac{\alpha\xi\left(A+mK_\alpha^2\right)\left(A+mK_\xi^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\psi^0 \\
-\left(\frac{\beta\left(A+mK_\beta^2\right)}{A}-\frac{\alpha\zeta\left(A+mK_\alpha^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)\frac{\vartheta\left(A+mK_\vartheta^2\right)-\frac{\xi\zeta\left(A+mK_\xi^2\right)\left(A+mK_\zeta^2\right)}{\varepsilon\left(A+mK_e^2\right)}}{\gamma\left(A+mK_g^2\right)-\frac{\zeta^2\left(A+mK_\zeta^2\right)^2}{\varepsilon\left(A+mK_e^2\right)}}\psi^0 \\
+\left(\frac{\eta\left(A+mK_f^2\right)}{A}-\frac{\xi^2\left(A+mK_\xi^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}-\frac{\left(\frac{\vartheta\left(A+mK_\vartheta^2\right)}{A}-\frac{\xi\zeta\left(A+mK_\xi^2\right)\left(A+mK_\zeta^2\right)}{A\varepsilon\left(A+mK_e^2\right)}\right)^2}{\frac{\gamma\left(A+mK_g^2\right)}{A}-\frac{\zeta^2\left(A+mK_\zeta^2\right)^2}{A\varepsilon\left(A+mK_e^2\right)}}\right)\Psi^0
\end{cases}
\tag{51}
$$

Using the first of two equations in (51), the first pair of the eigenvectors can be composed. Therefore, they read:

$$
\begin{pmatrix} U^{0(1)} \\ \varphi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix}=
\begin{pmatrix} U^{0(3)} \\ \varphi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix}=
\begin{pmatrix} U^{0(5)} \\ \varphi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix}=
\begin{pmatrix} U^{0(7)} \\ \varphi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix}=
\begin{pmatrix}
U^0=0 \\
\varphi^0=-\frac{\zeta}{\varepsilon}\Phi^0-\frac{\alpha}{\varepsilon}\psi^0-\frac{\xi}{\varepsilon}\Psi^0 \\
\psi^0=\lambda-\frac{\alpha\xi}{\varepsilon}-\frac{\left(\beta-\frac{\alpha\zeta}{\varepsilon}\right)\left(\vartheta-\frac{\xi\zeta}{\varepsilon}\right)}{\gamma-\frac{\zeta^2}{\varepsilon}} \\
\Phi^0=-\frac{\beta-\frac{\alpha\zeta}{\varepsilon}}{\gamma-\frac{\zeta^2}{\varepsilon}}\psi^0-\frac{\vartheta-\frac{\xi\zeta}{\varepsilon}}{\gamma-\frac{\zeta^2}{\varepsilon}}\Psi^0 \\
\Psi^0=-\mu+\frac{\alpha^2}{\varepsilon}+\frac{\left(\beta-\frac{\alpha\zeta}{\varepsilon}\right)^2}{\gamma-\frac{\zeta^2}{\varepsilon}}
\end{pmatrix}
\tag{52}
$$

$$
\begin{pmatrix} U^{0(9)} \\ \varphi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix}=
\begin{pmatrix}
U^0=\left(e\varphi^0+h\psi^0+g\Phi^0+f\Psi^0\right)/CK_{emgc}^2 \\
\varphi^0=-\frac{\zeta K_Z}{\varepsilon K_E}\Phi^0-\frac{\alpha K_A}{\varepsilon K_E}\psi^0-\frac{\xi K_S}{\varepsilon K_E}\Psi^0 \\
\psi^0=\frac{\lambda K_L}{K_{emgc}^2}-\frac{\alpha\xi K_A K_S}{\varepsilon K_E K_{emgc}^2}-\frac{\left(\frac{\beta K_B}{K_{emgc}^2}-\frac{\alpha\zeta K_A K_Z}{\varepsilon K_E K_{emgc}^2}\right)\left(\vartheta K_T-\frac{\xi\zeta K_S K_Z}{\varepsilon K_E}\right)}{\gamma K_G-\frac{\zeta^2 K_Z{}^2}{\varepsilon K_E}} \\
\Phi^0=-\frac{\beta K_B-\frac{\alpha\zeta K_A K_Z}{\varepsilon K_E}}{\gamma K_G-\frac{\zeta^2 K_Z{}^2}{\varepsilon K_E}}\psi^0-\frac{\vartheta K_T-\frac{\xi\zeta K_S K_Z}{\varepsilon K_E}}{\gamma K_G-\frac{\zeta^2 K_Z{}^2}{\varepsilon K_E}}\Psi^0 \\
\Psi^0=-\frac{\mu K_M}{K_{emgc}^2}+\frac{\alpha^2 K_A{}^2}{\varepsilon K_E K_{emgc}^2}+\frac{\left(\frac{\beta K_B}{K_{emgc}^2}-\frac{\alpha\zeta K_A K_Z}{\varepsilon K_E K_{emgc}^2}\right)^2}{\frac{\gamma K_G}{K_{emgc}^2}-\frac{\zeta^2 K_Z{}^2}{\varepsilon K_E K_{emgc}^2}}
\end{pmatrix}
\tag{53}
$$

The utilization of the second equation in (51) results in the second pair of the eigenvectors. They can be naturally written down as follows:

$$\begin{pmatrix} U^{0(1)} \\ \varphi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \varphi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \varphi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(7)} \\ \varphi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^0 = 0 \\ \varphi^0 = -\frac{\zeta}{\varepsilon}\Phi^0 - \frac{\alpha}{\varepsilon}\psi^0 - \frac{\xi}{\varepsilon}\Psi^0 \\ \psi^0 = -\eta + \frac{\xi^2}{\varepsilon} + \frac{\left(\vartheta - \frac{\xi\zeta}{\varepsilon}\right)^2}{\gamma - \frac{\zeta^2}{\varepsilon}} \\ \Phi^0 = -\frac{\beta - \frac{\alpha\zeta}{\varepsilon}}{\gamma - \frac{\zeta^2}{\varepsilon}}\psi^0 - \frac{\vartheta - \frac{\xi\zeta}{\varepsilon}}{\gamma - \frac{\zeta^2}{\varepsilon}}\Psi^0 \\ \Psi^0 = \lambda - \frac{\alpha\xi}{\varepsilon} - \frac{\left(\beta - \frac{\alpha\zeta}{\varepsilon}\right)\left(\vartheta - \frac{\xi\zeta}{\varepsilon}\right)}{\gamma - \frac{\zeta^2}{\varepsilon}} \end{pmatrix} \tag{54}$$

$$\begin{pmatrix} U^{0(9)} \\ \varphi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^0 = \left(e\varphi^0 + h\psi^0 + g\Phi^0 + f\Psi^0\right)/CK_{emgc}^2 \\ \varphi^0 = -\frac{\zeta K_Z}{\varepsilon K_E}\Phi^0 - \frac{\alpha K_A}{\varepsilon K_E}\psi^0 - \frac{\xi K_S}{\varepsilon K_E}\Psi^0 \\ \psi^0 = -\frac{\eta K_F}{K_{emgc}^2} + \frac{\xi^2 K_S{}^2}{\varepsilon K_E K_{emgc}^2} + \frac{\left(\frac{\vartheta K_T}{K_{emgc}^2} - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E K_{emgc}^2}\right)^2}{\frac{\gamma K_G}{K_{emgc}^2} - \frac{\zeta^2 K_Z{}^2}{\varepsilon K_E K_{emgc}^2}} \\ \Phi^0 = -\frac{\beta K_B - \frac{\alpha\zeta K_A K_Z}{\varepsilon K_E}}{\gamma K_G - \frac{\zeta^2 K_Z{}^2}{\varepsilon K_E}}\psi^0 - \frac{\vartheta K_T - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E}}{\gamma K_G - \frac{\zeta^2 K_Z{}^2}{\varepsilon K_E}}\Psi^0 \\ \Psi^0 = \frac{\lambda K_L}{K_{emgc}^2} - \frac{\alpha\xi K_A K_S}{\varepsilon K_E K_{emgc}^2} - \frac{\left(\frac{\beta K_B}{K_{emgc}^2} - \frac{\alpha\zeta K_A K_Z}{\varepsilon K_E K_{emgc}^2}\right)\left(\vartheta K_T - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E}\right)}{\gamma K_G - \frac{\zeta^2 K_Z{}^2}{\varepsilon K_E}} \end{pmatrix} \tag{55}$$

The third case

For the third case, the new regrouped set of four homogeneous equations (47) can be introduced as follows:

$$\begin{cases} \varepsilon\left(1 + mK_e^2/A\right)\varphi^0 + \xi\left(1 + mK_\xi^2/A\right)\Psi^0 \\ + \alpha\left(1 + mK_\alpha^2/A\right)\psi^0 + \zeta\left(1 + mK_\zeta^2/A\right)\Phi^0 = 0 \\ \xi\left(1 + mK_\xi^2/A\right)\varphi^0 + \eta\left(1 + mK_f^2/A\right)\Psi^0 \\ + \lambda\left(1 + mK_\lambda^2/A\right)\psi^0 + \vartheta\left(1 + mK_\vartheta^2/A\right)\Phi^0 = 0 \\ \alpha\left(1 + mK_\alpha^2/A\right)\varphi^0 + \lambda\left(1 + mK_\lambda^2/A\right)\Psi^0 \\ + \mu\left(1 + mK_m^2/A\right)\psi^0 + \beta\left(1 + mK_\beta^2/A\right)\Phi^0 = 0 \\ \zeta\left(1 + mK_\zeta^2/A\right)\varphi^0 + \vartheta\left(1 + mK_\vartheta^2/A\right)\Psi^0 \\ + \beta\left(1 + mK_\beta^2/A\right)\psi^0 + \gamma\left(1 + mK_g^2/A\right)\Phi^0 = 0 \end{cases} \tag{56}$$

Indeed, the fist equation in set (56) defines the eigenvector component φ^0 as a function of the rest components Ψ^0, ψ^0, and Φ^0. This dependence reads:

$$\varphi^0 = -\frac{\xi\left(A + mK_\xi^2\right)}{\varepsilon\left(A + mK_e^2\right)}\Psi^0 - \frac{\alpha\left(A + mK_\alpha^2\right)}{\varepsilon\left(A + mK_e^2\right)}\psi^0 - \frac{\zeta\left(A + mK_\zeta^2\right)}{\varepsilon\left(A + mK_e^2\right)}\Phi^0 \tag{57}$$

Using definition (57) for the standard mathematical procedure, the unknown φ^0 can be excluded for the further treatment and therefore, one can deal already with the following reduced set of three homogeneous equations:

$$\begin{cases} \left(\frac{\eta\left(A + mK_f^2\right)}{A} - \frac{\xi^2\left(A + mK_\xi^2\right)^2}{A\varepsilon\left(A + mK_e^2\right)}\right)\Psi^0 + \left(\frac{\lambda\left(A + mK_\lambda^2\right)}{A} - \frac{\alpha\xi\left(A + mK_\alpha^2\right)\left(A + mK_\xi^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)\psi^0 \\ + \left(\frac{\vartheta\left(A + mK_\vartheta^2\right)}{A} - \frac{\xi\zeta\left(A + mK_\xi^2\right)\left(A + mK_\zeta^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)\Phi^0 = 0 \\ \left(\frac{\lambda\left(A + mK_\lambda^2\right)}{A} - \frac{\alpha\xi\left(A + mK_\alpha^2\right)\left(A + mK_\xi^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)\Psi^0 + \left(\frac{\mu\left(A + mK_m^2\right)}{A} - \frac{\alpha^2\left(A + mK_\alpha^2\right)^2}{A\varepsilon\left(A + mK_e^2\right)}\right)\psi^0 \\ + \left(\frac{\beta\left(A + mK_\beta^2\right)}{A} - \frac{\alpha\zeta\left(A + mK_\alpha^2\right)\left(A + mK_\zeta^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)\Phi^0 = 0 \\ \left(\frac{\vartheta\left(A + mK_\vartheta^2\right)}{A} - \frac{\xi\zeta\left(A + mK_\xi^2\right)\left(A + mK_\zeta^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)\Psi^0 + \left(\frac{\beta\left(A + mK_\beta^2\right)}{A} - \frac{\alpha\zeta\left(A + mK_\alpha^2\right)\left(A + mK_\zeta^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)\psi^0 \\ + \left(\frac{\gamma\left(A + mK_g^2\right)}{A} - \frac{\zeta^2\left(A + mK_\zeta^2\right)^2}{A\varepsilon\left(A + mK_e^2\right)}\right)\Phi^0 = 0 \end{cases} \tag{58}$$

Analogically, the first equation in set (58) defines the eigenvector component Ψ^0 by

$$\Psi^0 = -\frac{\lambda\left(A + mK_\lambda^2\right) - \frac{\alpha\xi\left(A + mK_\alpha^2\right)\left(A + mK_\xi^2\right)}{\varepsilon\left(A + mK_e^2\right)}}{\eta\left(A + mK_f^2\right) - \frac{\xi^2\left(A + mK_\xi^2\right)^2}{\varepsilon\left(A + mK_e^2\right)}}\psi^0 - \frac{\vartheta\left(A + mK_\vartheta^2\right) - \frac{\xi\zeta\left(A + mK_\xi^2\right)\left(A + mK_\zeta^2\right)}{\varepsilon\left(A + mK_e^2\right)}}{\eta\left(A + mK_f^2\right) - \frac{\xi^2\left(A + mK_\xi^2\right)^2}{\varepsilon\left(A + mK_e^2\right)}}\Phi^0 \tag{59}$$

The application of definition (59) for substitution in the second and third equations in set (58) can provide the final set of two homogeneous equations. These two equations can be exposed in the following form:

$$\begin{cases} \left(\frac{\mu\left(A + mK_m^2\right)}{A} - \frac{\alpha^2\left(A + mK_\alpha^2\right)^2}{A\varepsilon\left(A + mK_e^2\right)} - \frac{\left(\frac{\lambda\left(A + mK_\lambda^2\right)}{A} - \frac{\alpha\xi\left(A + mK_\alpha^2\right)\left(A + mK_\xi^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)^2}{\frac{\eta\left(A + mK_f^2\right)}{A} - \frac{\xi^2\left(A + mK_\xi^2\right)^2}{A\varepsilon\left(A + mK_e^2\right)}}\right)\psi^0 \\ - \left(\frac{\lambda\left(A + mK_\lambda^2\right)}{A} - \frac{\alpha\xi\left(A + mK_\alpha^2\right)\left(A + mK_\xi^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)\frac{\vartheta\left(A + mK_\vartheta^2\right) - \frac{\xi\zeta\left(A + mK_\xi^2\right)\left(A + mK_\zeta^2\right)}{\varepsilon\left(A + mK_e^2\right)}}{\eta\left(A + mK_f^2\right) - \frac{\xi^2\left(A + mK_\xi^2\right)^2}{\varepsilon\left(A + mK_e^2\right)}}\Phi^0 \\ + \left(\frac{\beta\left(A + mK_\beta^2\right)}{A} - \frac{\alpha\zeta\left(A + mK_\alpha^2\right)\left(A + mK_\zeta^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)\Phi^0 = 0 \\ - \left(\frac{\lambda\left(A + mK_\lambda^2\right)}{A} - \frac{\alpha\xi\left(A + mK_\alpha^2\right)\left(A + mK_\xi^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)\frac{\vartheta\left(A + mK_\vartheta^2\right) - \frac{\xi\zeta\left(A + mK_\xi^2\right)\left(A + mK_\zeta^2\right)}{\varepsilon\left(A + mK_e^2\right)}}{\eta\left(A + mK_f^2\right) - \frac{\xi^2\left(A + mK_\xi^2\right)^2}{\varepsilon\left(A + mK_e^2\right)}}\psi^0 \\ + \left(\frac{\beta\left(A + mK_\beta^2\right)}{A} - \frac{\alpha\zeta\left(A + mK_\alpha^2\right)\left(A + mK_\zeta^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)\psi^0 \\ + \left(\frac{\gamma\left(A + mK_g^2\right)}{A} - \frac{\zeta^2\left(A + mK_\zeta^2\right)^2}{A\varepsilon\left(A + mK_e^2\right)} - \frac{\left(\frac{\vartheta\left(A + mK_\vartheta^2\right)}{A} - \frac{\xi\zeta\left(A + mK_\xi^2\right)\left(A + mK_\zeta^2\right)}{A\varepsilon\left(A + mK_e^2\right)}\right)^2}{\frac{\eta\left(A + mK_f^2\right)}{A} - \frac{\xi^2\left(A + mK_\xi^2\right)^2}{A\varepsilon\left(A + mK_e^2\right)}}\right)\Phi^0 = 0 \end{cases} \tag{60}$$

Using the first equation in set (60), the first eigenvectors read:

$$\begin{pmatrix} U^{0(1)} \\ \varphi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \varphi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \varphi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(7)} \\ \varphi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^0 = 0 \\ \varphi^0 = -\frac{\xi}{\varepsilon}\Psi^0 - \frac{\alpha}{\varepsilon}\psi^0 - \frac{\zeta}{\varepsilon}\Phi^0 \\ \psi^0 = \beta - \frac{\alpha\zeta}{\varepsilon} - \frac{\left(\lambda - \frac{\alpha\xi}{\varepsilon}\right)\left(\vartheta - \frac{\xi\zeta}{\varepsilon}\right)}{\eta - \frac{\xi^2}{\varepsilon}} \\ \Phi^0 = -\mu + \frac{\alpha^2}{\varepsilon} + \frac{\left(\lambda - \frac{\alpha\xi}{\varepsilon}\right)^2}{\eta - \frac{\xi^2}{\varepsilon}} \\ \Psi^0 = -\frac{\lambda - \frac{\alpha\xi}{\varepsilon}}{\eta - \frac{\xi^2}{\varepsilon}}\psi^0 - \frac{\vartheta - \frac{\xi\zeta}{\varepsilon}}{\eta - \frac{\xi^2}{\varepsilon}}\Phi^0 \end{pmatrix} \quad (61)$$

$$\begin{pmatrix} U^{0(9)} \\ \varphi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^0 = \left(e\varphi^0 + h\psi^0 + g\Phi^0 + f\Psi^0\right)/CK_{emgc}^2 \\ \varphi^0 = -\frac{\xi K_S}{\varepsilon K_E}\Psi^0 - \frac{\alpha K_A}{\varepsilon K_E}\psi^0 - \frac{\zeta K_Z}{\varepsilon K_E}\Phi^0 \\ \psi^0 = \frac{\beta K_B}{K_{emgc}^2} - \frac{\alpha\zeta K_A K_Z}{\varepsilon K_E K_{emgc}^2} - \left(\frac{\lambda K_L}{K_{emgc}^2} - \frac{\alpha\xi K_A K_S}{\varepsilon K_E K_{emgc}^2}\right)\frac{\vartheta K_T - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E}}{\eta K_F - \frac{\xi^2 K_S^2}{\varepsilon K_E}} \\ \Phi^0 = -\frac{\mu K_M}{K_{emgc}^2} + \frac{\alpha^2 K_A^2}{\varepsilon K_E K_{emgc}^2} + \frac{\left(\frac{\lambda K_L}{K_{emgc}^2} - \frac{\alpha\xi K_A K_S}{\varepsilon K_E K_{emgc}^2}\right)^2}{\frac{\eta K_F}{K_{emgc}^2} - \frac{\xi^2 K_S^2}{\varepsilon K_E K_{emgc}^2}} \\ \Psi^0 = -\frac{\lambda K_L - \frac{\alpha\xi K_A K_S}{\varepsilon K_E}}{\eta K_F - \frac{\xi^2 K_S^2}{\varepsilon K_E}}\psi^0 - \frac{\vartheta K_T - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E}}{\eta K_F - \frac{\xi^2 K_S^2}{\varepsilon K_E}}\Phi^0 \end{pmatrix} \quad (62)$$

The second eigenvectors can be obtained by the use of the second equation in set (60). Their components can be written down as follows:

$$\begin{pmatrix} U^{0(1)} \\ \varphi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \varphi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \varphi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(7)} \\ \varphi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^0 = 0 \\ \varphi^0 = -\frac{\xi}{\varepsilon}\Psi^0 - \frac{\alpha}{\varepsilon}\psi^0 - \frac{\zeta}{\varepsilon}\Phi^0 \\ \psi^0 = -\gamma + \frac{\zeta^2}{\varepsilon} + \frac{\left(\vartheta - \frac{\xi\zeta}{\varepsilon}\right)^2}{\eta - \frac{\xi^2}{\varepsilon}} \\ \Phi^0 = \beta - \frac{\alpha\zeta}{\varepsilon} - \frac{\left(\lambda - \frac{\alpha\xi}{\varepsilon}\right)\left(\vartheta - \frac{\xi\zeta}{\varepsilon}\right)}{\eta - \frac{\xi^2}{\varepsilon}} \\ \Psi^0 = -\frac{\lambda - \frac{\alpha\xi}{\varepsilon}}{\eta - \frac{\xi^2}{\varepsilon}}\psi^0 - \frac{\vartheta - \frac{\xi\zeta}{\varepsilon}}{\eta - \frac{\xi^2}{\varepsilon}}\Phi^0 \end{pmatrix} \quad (63)$$

$$\begin{pmatrix} U^{0(9)} \\ \varphi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^0 = \left(e\varphi^0 + h\psi^0 + g\Phi^0 + f\Psi^0\right)/CK_{emgc}^2 \\ \varphi^0 = -\frac{\xi K_S}{\varepsilon K_E}\Psi^0 - \frac{\alpha K_A}{\varepsilon K_E}\psi^0 - \frac{\zeta K_Z}{\varepsilon K_E}\Phi^0 \\ \psi^0 = -\frac{\gamma K_G}{K_{emgc}^2} + \frac{\zeta^2 K_Z^2}{\varepsilon K_E K_{emgc}^2} + \frac{\left(\frac{\vartheta K_T}{K_{emgc}^2} - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E K_{emgc}^2}\right)^2}{\frac{\eta K_F}{K_{emgc}^2} - \frac{\xi^2 K_S^2}{\varepsilon K_E K_{emgc}^2}} \\ \Phi^0 = \frac{\beta K_B}{K_{emgc}^2} - \frac{\alpha\zeta K_A K_Z}{\varepsilon K_E K_{emgc}^2} - \left(\frac{\lambda K_L}{K_{emgc}^2} - \frac{\alpha\xi K_A K_S}{\varepsilon K_E K_{emgc}^2}\right)\frac{\vartheta K_T - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E}}{\eta K_F - \frac{\xi^2 K_S^2}{\varepsilon K_E}} \\ \Psi^0 = -\frac{\lambda K_L - \frac{\alpha\xi K_A K_S}{\varepsilon K_E}}{\eta K_F - \frac{\xi^2 K_S^2}{\varepsilon K_E}}\psi^0 - \frac{\vartheta K_T - \frac{\xi\zeta K_S K_Z}{\varepsilon K_E}}{\eta K_F - \frac{\xi^2 K_S^2}{\varepsilon K_E}}\Phi^0 \end{pmatrix} \quad (64)$$

The fourth case

In order to obtain the other possible forms of the eigenvectors it is possible to treat the following order of equations (47):

$$\begin{cases} \mu\left(1 + mK_m^2/A\right)\psi^0 + \beta\left(1 + mK_\beta^2/A\right)\Phi^0 \\ + \alpha\left(1 + mK_\alpha^2/A\right)\varphi^0 + \lambda\left(1 + mK_\lambda^2/A\right)\Psi^0 = 0 \\ \beta\left(1 + mK_\beta^2/A\right)\psi^0 + \gamma\left(1 + mK_g^2/A\right)\Phi^0 \\ + \zeta\left(1 + mK_\zeta^2/A\right)\varphi^0 + \vartheta\left(1 + mK_\vartheta^2/A\right)\Psi^0 = 0 \\ \alpha\left(1 + mK_\alpha^2/A\right)\psi^0 + \zeta\left(1 + mK_\zeta^2/A\right)\Phi^0 \\ + \varepsilon\left(1 + mK_e^2/A\right)\varphi^0 + \xi\left(1 + mK_\xi^2/A\right)\Psi^0 = 0 \\ \lambda\left(1 + mK_\lambda^2/A\right)\psi^0 + \vartheta\left(1 + mK_\vartheta^2/A\right)\Phi^0 \\ + \xi\left(1 + mK_\xi^2/A\right)\varphi^0 + \eta\left(1 + mK_f^2/A\right)\Psi^0 = 0 \end{cases} \quad (65)$$

It is natural to exploit the first equation in set (65) to determine the eigenvector component ψ^0 as a function of the components Φ^0, φ^0, and Ψ^0. Therefore, this dependence can be introduced as follows:

$$\psi^0 = -\frac{\beta\left(A + mK_\beta^2\right)}{\mu\left(A + mK_m^2\right)}\Phi^0 - \frac{\alpha\left(A + mK_\alpha^2\right)}{\mu\left(A + mK_m^2\right)}\varphi^0 - \frac{\lambda\left(A + mK_\lambda^2\right)}{\mu\left(A + mK_m^2\right)}\Psi^0 \quad (66)$$

A substitution of definition (66) into equations (65), but the first equation in set (65), leads to the homogeneous set of three equations in three unknowns representing the eigenvector components Φ^0, φ^0, and Ψ^0. These three complicated equations are inscribed as follows:

$$\begin{cases}\left(\frac{\gamma\left(A+mK_g^2\right)}{A}-\frac{\beta^2\left(A+mK_\beta^2\right)^2}{A\mu\left(A+mK_m^2\right)}\right)\mathit{\Phi}^0+\left(\frac{\zeta\left(A+mK_\zeta^2\right)}{A}-\frac{\alpha\beta\left(A+mK_\alpha^2\right)\left(A+mK_\beta^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\varphi^0 \\ +\left(\frac{\vartheta\left(A+mK_\vartheta^2\right)}{A}-\frac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\mathit{\Psi}^0=0 \\ \left(\frac{\zeta\left(A+mK_\zeta^2\right)}{A}-\frac{\alpha\beta\left(A+mK_\alpha^2\right)\left(A+mK_\beta^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\mathit{\Phi}^0+\left(\frac{\varepsilon\left(A+mK_e^2\right)}{A}-\frac{\alpha^2\left(A+mK_\alpha^2\right)^2}{A\mu\left(A+mK_m^2\right)}\right)\varphi^0 \\ +\left(\frac{\xi\left(A+mK_\xi^2\right)}{A}-\frac{\alpha\lambda\left(A+mK_\alpha^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\mathit{\Psi}^0=0 \\ \left(\frac{\vartheta\left(A+mK_\vartheta^2\right)}{A}-\frac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\mathit{\Phi}^0+\left(\frac{\xi\left(A+mK_\xi^2\right)}{A}-\frac{\alpha\lambda\left(A+mK_\alpha^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\varphi^0 \\ +\left(\frac{\eta\left(A+mK_f^2\right)}{A}-\frac{\lambda^2\left(A+mK_\lambda^2\right)^2}{A\mu\left(A+mK_m^2\right)}\right)\mathit{\Psi}^0=0\end{cases} \tag{67}$$

Next, let's use the first equation in set (67) for definition of the eigenvector component $\mathit{\Phi}^0$. Thus, it is defined by

$$\mathit{\Phi}^0=-\frac{\zeta\left(A+mK_\zeta^2\right)-\frac{\alpha\beta\left(A+mK_\alpha^2\right)\left(A+mK_\beta^2\right)}{\mu\left(A+mK_m^2\right)}}{\gamma\left(A+mK_g^2\right)-\frac{\beta^2\left(A+mK_\beta^2\right)^2}{\mu\left(A+mK_m^2\right)}}\varphi^0-\frac{\vartheta\left(A+mK_\vartheta^2\right)-\frac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{\mu\left(A+mK_m^2\right)}}{\gamma\left(A+mK_g^2\right)-\frac{\beta^2\left(A+mK_\beta^2\right)^2}{\mu\left(A+mK_m^2\right)}}\mathit{\Psi}^0 \tag{68}$$

Two homogeneous equations can be finally written and used for determination of the eigenvector components φ^0 and $\mathit{\Psi}^0$. These final equations are

$$\begin{cases}0=\left(\frac{\varepsilon\left(A+mK_e^2\right)}{A}-\frac{\alpha^2\left(A+mK_\alpha^2\right)^2}{A\mu\left(A+mK_m^2\right)}-\frac{\left(\frac{\zeta\left(A+mK_\zeta^2\right)}{A}-\frac{\alpha\beta\left(A+mK_\alpha^2\right)\left(A+mK_\beta^2\right)}{A\mu\left(A+mK_m^2\right)}\right)^2}{\frac{\gamma\left(A+mK_g^2\right)}{A}-\frac{\beta^2\left(A+mK_\beta^2\right)^2}{A\mu\left(A+mK_m^2\right)}}\right)\varphi^0 \\ +\left(\frac{\xi\left(A+mK_\xi^2\right)}{A}-\frac{\alpha\lambda\left(A+mK_\alpha^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\mathit{\Psi}^0 \\ -\left(\frac{\zeta\left(A+mK_\zeta^2\right)}{A}-\frac{\alpha\beta\left(A+mK_\alpha^2\right)\left(A+mK_\beta^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\frac{\vartheta\left(A+mK_\vartheta^2\right)-\frac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{\mu\left(A+mK_m^2\right)}}{\gamma\left(A+mK_g^2\right)-\frac{\beta^2\left(A+mK_\beta^2\right)^2}{\mu\left(A+mK_m^2\right)}}\mathit{\Psi}^0 \\ 0=\left(\frac{\xi\left(A+mK_\xi^2\right)}{A}-\frac{\alpha\lambda\left(A+mK_\alpha^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\varphi^0 \\ -\left(\frac{\zeta\left(A+mK_\zeta^2\right)}{A}-\frac{\alpha\beta\left(A+mK_\alpha^2\right)\left(A+mK_\beta^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\frac{\vartheta\left(A+mK_\vartheta^2\right)-\frac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{\mu\left(A+mK_m^2\right)}}{\gamma\left(A+mK_g^2\right)-\frac{\beta^2\left(A+mK_\beta^2\right)^2}{\mu\left(A+mK_m^2\right)}}\varphi^0 \\ +\left(\frac{\eta\left(A+mK_f^2\right)}{A}-\frac{\lambda^2\left(A+mK_\lambda^2\right)^2}{A\mu\left(A+mK_m^2\right)}-\frac{\left(\frac{\vartheta\left(A+mK_\vartheta^2\right)}{A}-\frac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)^2}{\frac{\gamma\left(A+mK_g^2\right)}{A}-\frac{\beta^2\left(A+mK_\beta^2\right)^2}{A\mu\left(A+mK_m^2\right)}}\right)\mathit{\Psi}^0\end{cases} \tag{69}$$

The utilization of the first of two equations (69) provides the following first eigenvectors:

$$\begin{pmatrix}U^{0(1)}\\ \varphi^{0(1)}\\ \psi^{0(1)}\\ \mathit{\Phi}^{0(1)}\\ \mathit{\Psi}^{0(1)}\end{pmatrix}=\begin{pmatrix}U^{0(3)}\\ \varphi^{0(3)}\\ \psi^{0(3)}\\ \mathit{\Phi}^{0(3)}\\ \mathit{\Psi}^{0(3)}\end{pmatrix}=\begin{pmatrix}U^{0(5)}\\ \varphi^{0(5)}\\ \psi^{0(5)}\\ \mathit{\Phi}^{0(5)}\\ \mathit{\Psi}^{0(5)}\end{pmatrix}=\begin{pmatrix}U^{0(7)}\\ \varphi^{0(7)}\\ \psi^{0(7)}\\ \mathit{\Phi}^{0(7)}\\ \mathit{\Psi}^{0(7)}\end{pmatrix}=\begin{pmatrix}U^0=0\\ \varphi^0=\xi-\frac{\alpha\lambda}{\mu}-\frac{\left(\zeta-\frac{\alpha\beta}{\mu}\right)\left(\vartheta-\frac{\beta\lambda}{\mu}\right)}{\gamma-\frac{\beta^2}{\mu}}\\ \psi^0=-\frac{\beta}{\mu}\mathit{\Phi}^0-\frac{\alpha}{\mu}\varphi^0-\frac{\lambda}{\mu}\mathit{\Psi}^0\\ \mathit{\Phi}^0=-\frac{\zeta-\frac{\alpha\beta}{\mu}}{\gamma-\frac{\beta^2}{\mu}}\varphi^0-\frac{\vartheta-\frac{\beta\lambda}{\mu}}{\gamma-\frac{\beta^2}{\mu}}\mathit{\Psi}^0\\ \mathit{\Psi}^0=-\varepsilon+\frac{\alpha^2}{\mu}+\frac{\left(\zeta-\frac{\alpha\beta}{\mu}\right)^2}{\gamma-\frac{\beta^2}{\mu}}\end{pmatrix} \tag{70}$$

$$\begin{pmatrix}U^{0(9)}\\ \varphi^{0(9)}\\ \psi^{0(9)}\\ \mathit{\Phi}^{0(9)}\\ \mathit{\Psi}^{0(9)}\end{pmatrix}=\begin{pmatrix}U^0=\left(e\varphi^0+h\psi^0+g\mathit{\Phi}^0+f\mathit{\Psi}^0\right)/CK_{emgc}^2\\ \varphi^0=\frac{\xi K_S}{K_{emgc}^2}-\frac{\alpha\lambda K_A K_L}{\mu K_M K_{emgc}^2}-\left(\frac{\zeta K_Z}{K_{emgc}^2}-\frac{\alpha\beta K_A K_B}{\mu K_M K_{emgc}^2}\right)\frac{\vartheta K_T-\frac{\beta\lambda K_B K_L}{\mu K_M}}{\gamma K_G-\frac{\beta^2 K_B{}^2}{\mu K_M}}\\ \psi^0=-\frac{\beta K_B}{\mu K_M}\mathit{\Phi}^0-\frac{\alpha K_A}{\mu K_M}\varphi^0-\frac{\lambda K_L}{\mu K_M}\mathit{\Psi}^0\\ \mathit{\Phi}^0=-\frac{\zeta K_Z-\frac{\alpha\beta K_A K_B}{\mu K_M}}{\gamma K_G-\frac{\beta^2 K_B{}^2}{\mu K_M}}\varphi^0-\frac{\vartheta K_T-\frac{\beta\lambda K_B K_L}{\mu K_M}}{\gamma K_G-\frac{\beta^2 K_B{}^2}{\mu K_M}}\mathit{\Psi}^0\\ \mathit{\Psi}^0=-\frac{\varepsilon K_E}{K_{emgc}^2}+\frac{\alpha^2 K_A{}^2}{\mu K_M K_{emgc}^2}+\frac{\left(\frac{\zeta K_Z}{K_{emgc}^2}-\frac{\alpha\beta K_A K_B}{\mu K_M K_{emgc}^2}\right)^2}{\frac{\gamma K_G}{K_{emgc}^2}-\frac{\beta^2 K_B{}^2}{\mu K_M K_{emgc}^2}}\end{pmatrix} \tag{71}$$

The second eigenvectors for this case can be obtained by using the second equation in set (69). They are defined by

$$\begin{pmatrix}U^{0(1)}\\ \varphi^{0(1)}\\ \psi^{0(1)}\\ \mathit{\Phi}^{0(1)}\\ \mathit{\Psi}^{0(1)}\end{pmatrix}=\begin{pmatrix}U^{0(3)}\\ \varphi^{0(3)}\\ \psi^{0(3)}\\ \mathit{\Phi}^{0(3)}\\ \mathit{\Psi}^{0(3)}\end{pmatrix}=\begin{pmatrix}U^{0(5)}\\ \varphi^{0(5)}\\ \psi^{0(5)}\\ \mathit{\Phi}^{0(5)}\\ \mathit{\Psi}^{0(5)}\end{pmatrix}=\begin{pmatrix}U^{0(7)}\\ \varphi^{0(7)}\\ \psi^{0(7)}\\ \mathit{\Phi}^{0(7)}\\ \mathit{\Psi}^{0(7)}\end{pmatrix}$$

$$=\begin{pmatrix}U^0=0\\ \varphi^0=-\eta+\frac{\lambda^2}{\mu}+\frac{\left(\vartheta-\frac{\beta\lambda}{\mu}\right)^2}{\gamma-\frac{\beta^2}{\mu}}\\ \psi^0=-\frac{\beta}{\mu}\mathit{\Phi}^0-\frac{\alpha}{\mu}\varphi^0-\frac{\lambda}{\mu}\mathit{\Psi}^0\\ \mathit{\Phi}^0=-\frac{\zeta-\frac{\alpha\beta}{\mu}}{\gamma-\frac{\beta^2}{\mu}}\varphi^0-\frac{\vartheta-\frac{\beta\lambda}{\mu}}{\gamma-\frac{\beta^2}{\mu}}\mathit{\Psi}^0\\ \mathit{\Psi}^0=\xi-\frac{\alpha\lambda}{\mu}-\frac{\left(\zeta-\frac{\alpha\beta}{\mu}\right)\left(\vartheta-\frac{\beta\lambda}{\mu}\right)}{\gamma-\frac{\beta^2}{\mu}}\end{pmatrix} \tag{72}$$

$$
\begin{pmatrix} U^{0(9)} \\ \varphi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \begin{pmatrix} U^0 = \left(e\varphi^0 + h\psi^0 + g\Phi^0 + f\Psi^0\right)/CK_{emgc}^2 \\ \varphi^0 = -\dfrac{\eta K_F}{K_{emgc}^2} + \dfrac{\lambda^2 K_L{}^2}{\mu K_M K_{emgc}^2} + \dfrac{\left(\dfrac{\vartheta K_T}{K_{emgc}^2} - \dfrac{\beta\lambda K_B K_L}{\mu K_M K_{emgc}^2}\right)^2}{\dfrac{\gamma K_G}{K_{emgc}^2} - \dfrac{\beta^2 K_B{}^2}{\mu K_M K_{emgc}^2}} \\ \psi^0 = -\dfrac{\beta K_B}{\mu K_M}\Phi^0 - \dfrac{\alpha K_A}{\mu K_M}\varphi^0 - \dfrac{\lambda K_L}{\mu K_M}\Psi^0 \\ \Phi^0 = -\dfrac{\zeta K_Z - \dfrac{\alpha\beta K_A K_B}{\mu K_M}}{\gamma K_G - \dfrac{\beta^2 K_B{}^2}{\mu K_M}}\varphi^0 - \dfrac{\vartheta K_T - \dfrac{\beta\lambda K_B K_L}{\mu K_M}}{\gamma K_G - \dfrac{\beta^2 K_B{}^2}{\mu K_M}}\Psi^0 \\ \Psi^0 = \dfrac{\xi K_S}{K_{emgc}^2} - \dfrac{\alpha\lambda K_A K_L}{\mu K_M K_{emgc}^2} - \left(\dfrac{\zeta K_Z}{K_{emgc}^2} - \dfrac{\alpha\beta K_A K_B}{\mu K_M K_{emgc}^2}\right)\dfrac{\vartheta K_T - \dfrac{\beta\lambda K_B K_L}{\mu K_M}}{\gamma K_G - \dfrac{\beta^2 K_B{}^2}{\mu K_M}} \end{pmatrix} \tag{73}
$$

The fifth case

Four homogeneous equations (47) can be also rewritten in the following order to get the other set of the eigenvector components:

$$
\begin{cases}
\mu\left(1+mK_m^2/A\right)\psi^0 + \lambda\left(1+mK_\lambda^2/A\right)\Psi^0 \\
+\alpha\left(1+mK_\alpha^2/A\right)\varphi^0 + \beta\left(1+mK_\beta^2/A\right)\Phi^0 = 0 \\
\lambda\left(1+mK_\lambda^2/A\right)\psi^0 + \eta\left(1+mK_f^2/A\right)\Psi^0 \\
+\xi\left(1+mK_\xi^2/A\right)\varphi^0 + \vartheta\left(1+mK_\vartheta^2/A\right)\Phi^0 = 0 \\
\alpha\left(1+mK_\alpha^2/A\right)\psi^0 + \xi\left(1+mK_\xi^2/A\right)\Psi^0 \\
+\varepsilon\left(1+mK_e^2/A\right)\varphi^0 + \zeta\left(1+mK_\zeta^2/A\right)\Phi^0 = 0 \\
\beta\left(1+mK_\beta^2/A\right)\psi^0 + \vartheta\left(1+mK_\vartheta^2/A\right)\Psi^0 \\
+\zeta\left(1+mK_\zeta^2/A\right)\varphi^0 + \gamma\left(1+mK_g^2/A\right)\Phi^0 = 0
\end{cases} \tag{74}
$$

As a result, the firs equation in set (74) can provide the eigenvector component ψ^0 as the following function of the eigenvector components Ψ^0, φ^0, and Φ^0:

$$
\psi^0 = -\frac{\lambda\left(A+mK_\lambda^2\right)}{\mu\left(A+mK_m^2\right)}\Psi^0 - \frac{\alpha\left(A+mK_\alpha^2\right)}{\mu\left(A+mK_m^2\right)}\varphi^0 - \frac{\beta\left(A+mK_\beta^2\right)}{\mu\left(A+mK_m^2\right)}\Phi^0 \tag{75}
$$

So, equation (75) must be used for set (74) to reduce it. The following set of three homogeneous equations can be then obtained:

$$
\begin{cases}
\left(\dfrac{\eta\left(A+mK_f^2\right)}{A} - \dfrac{\lambda^2\left(A+mK_\lambda^2\right)^2}{A\mu\left(A+mK_m^2\right)}\right)\Psi^0 + \left(\dfrac{\xi\left(A+mK_\xi^2\right)}{A} - \dfrac{\alpha\lambda\left(A+mK_\alpha^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\varphi^0 \\
+\left(\dfrac{\vartheta\left(A+mK_\vartheta^2\right)}{A} - \dfrac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\Phi^0 = 0 \\
\left(\dfrac{\xi\left(A+mK_\xi^2\right)}{A} - \dfrac{\alpha\lambda\left(A+mK_\alpha^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\Psi^0 + \left(\dfrac{\varepsilon\left(A+mK_e^2\right)}{A} - \dfrac{\alpha^2\left(A+mK_\alpha^2\right)^2}{A\mu\left(A+mK_m^2\right)}\right)\varphi^0 \\
+\left(\dfrac{\zeta\left(A+mK_\zeta^2\right)}{A} - \dfrac{\alpha\beta\left(A+mK_\alpha^2\right)\left(A+mK_\beta^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\Phi^0 = 0 \\
\left(\dfrac{\vartheta\left(A+mK_\vartheta^2\right)}{A} - \dfrac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\Psi^0 + \left(\dfrac{\zeta\left(A+mK_\zeta^2\right)}{A} - \dfrac{\alpha\beta\left(A+mK_\alpha^2\right)\left(A+mK_\beta^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\varphi^0 \\
+\left(\dfrac{\gamma\left(A+mK_g^2\right)}{A} - \dfrac{\beta^2\left(A+mK_\beta^2\right)^2}{A\mu\left(A+mK_m^2\right)}\right)\Phi^0 = 0
\end{cases} \tag{76}
$$

The first equation in set (76) defines the eigenvector component Ψ^0 as follows:

$$
\Psi^0 = -\frac{\xi\left(A+mK_\xi^2\right) - \dfrac{\alpha\lambda\left(A+mK_\alpha^2\right)\left(A+mK_\lambda^2\right)}{\mu\left(A+mK_m^2\right)}}{\eta\left(A+mK_f^2\right) - \dfrac{\lambda^2\left(A+mK_\lambda^2\right)^2}{\mu\left(A+mK_m^2\right)}}\varphi^0 - \frac{\vartheta\left(A+mK_\vartheta^2\right) - \dfrac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{\mu\left(A+mK_m^2\right)}}{\eta\left(A+mK_f^2\right) - \dfrac{\lambda^2\left(A+mK_\lambda^2\right)^2}{\mu\left(A+mK_m^2\right)}}\Phi^0 \tag{77}
$$

The deployment of definition (77) for set (76) leads to the following two final equations, with which it is already possible to obtain the values of the eigenvector components φ^0 and Φ^0 in explicit forms. These two final equations can be formed as follows:

$$
\begin{cases}
0 = \left(\dfrac{\varepsilon\left(A+mK_e^2\right)}{A} - \dfrac{\alpha^2\left(A+mK_\alpha^2\right)^2}{A\mu\left(A+mK_m^2\right)} - \dfrac{\left(\dfrac{\xi\left(A+mK_\xi^2\right)}{A} - \dfrac{\alpha\lambda\left(A+mK_\alpha^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)^2}{\dfrac{\eta\left(A+mK_f^2\right)}{A} - \dfrac{\lambda^2\left(A+mK_\lambda^2\right)^2}{A\mu\left(A+mK_m^2\right)}}\right)\varphi^0 \\
+\left(\dfrac{\zeta\left(A+mK_\zeta^2\right)}{A} - \dfrac{\alpha\beta\left(A+mK_\alpha^2\right)\left(A+mK_\beta^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\Phi^0 \\
-\left(\dfrac{\xi\left(A+mK_\xi^2\right)}{A} - \dfrac{\alpha\lambda\left(A+mK_\alpha^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\dfrac{\vartheta\left(A+mK_\vartheta^2\right) - \dfrac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{\mu\left(A+mK_m^2\right)}}{\eta\left(A+mK_f^2\right) - \dfrac{\lambda^2\left(A+mK_\lambda^2\right)^2}{\mu\left(A+mK_m^2\right)}}\Phi^0 \\
0 = \left(\dfrac{\zeta\left(A+mK_\zeta^2\right)}{A} - \dfrac{\alpha\beta\left(A+mK_\alpha^2\right)\left(A+mK_\beta^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\varphi^0 \\
-\left(\dfrac{\xi\left(A+mK_\xi^2\right)}{A} - \dfrac{\alpha\lambda\left(A+mK_\alpha^2\right)\left(A+mK_\lambda^2\right)}{A\mu\left(A+mK_m^2\right)}\right)\dfrac{\vartheta\left(A+mK_\vartheta^2\right) - \dfrac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{\mu\left(A+mK_m^2\right)}}{\eta\left(A+mK_f^2\right) - \dfrac{\lambda^2\left(A+mK_\lambda^2\right)^2}{\mu\left(A+mK_m^2\right)}}\varphi^0 \\
+\left(\dfrac{\gamma\left(A+mK_g^2\right)}{A} - \dfrac{\beta^2\left(A+mK_\beta^2\right)^2}{A\mu\left(A+mK_m^2\right)} - \dfrac{\left(\vartheta\left(A+mK_\vartheta^2\right) - \dfrac{\beta\lambda\left(A+mK_\beta^2\right)\left(A+mK_\lambda^2\right)}{\mu\left(A+mK_m^2\right)}\right)^2}{\eta\left(A+mK_f^2\right) - \dfrac{\lambda^2\left(A+mK_\lambda^2\right)^2}{\mu\left(A+mK_m^2\right)}}\right)\Phi^0
\end{cases} \tag{78}
$$

The first equation in set (78) determines the first eigenvectors. Their components take the following forms:

$$
\begin{pmatrix} U^{0(1)} \\ \varphi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \varphi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \varphi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(7)} \\ \varphi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \begin{pmatrix} U^0 = 0 \\ \varphi^0 = \zeta - \dfrac{\alpha\beta}{\mu} - \dfrac{\left(\xi - \dfrac{\alpha\lambda}{\mu}\right)\left(\vartheta - \dfrac{\beta\lambda}{\mu}\right)}{\eta - \dfrac{\lambda^2}{\mu}} \\ \psi^0 = -\dfrac{\lambda}{\mu}\Psi^0 - \dfrac{\alpha}{\mu}\varphi^0 - \dfrac{\beta}{\mu}\Phi^0 \\ \Phi^0 = -\varepsilon + \dfrac{\alpha^2}{\mu} + \dfrac{\left(\xi - \dfrac{\alpha\lambda}{\mu}\right)^2}{\eta - \dfrac{\lambda^2}{\mu}} \\ \Psi^0 = -\dfrac{\xi - \dfrac{\alpha\lambda}{\mu}}{\eta - \dfrac{\lambda^2}{\mu}}\varphi^0 - \dfrac{\vartheta - \dfrac{\beta\lambda}{\mu}}{\eta - \dfrac{\lambda^2}{\mu}}\Phi^0 \end{pmatrix} \tag{79}
$$

$$\begin{pmatrix} U^{0(9)} \\ \varphi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \left(\begin{array}{c} U^0 = \left(e\varphi^0 + h\psi^0 + g\Phi^0 + f\Psi^0\right) / CK_{emgc}^2 \\ \varphi^0 = \frac{\zeta K_Z}{K_{emgc}^2} - \frac{\alpha\beta K_A K_B}{\mu K_M K_{emgc}^2} - \left(\frac{\xi K_S}{K_{emgc}^2} - \frac{\alpha\lambda K_A K_L}{\mu K_M K_{emgc}^2} \right) \frac{\vartheta K_T - \frac{\beta\lambda K_B K_L}{\mu K_M}}{\eta K_F - \frac{\lambda^2 K_L^2}{\mu K_M}} \\ \psi^0 = -\frac{\lambda K_L}{\mu K_M}\Psi^0 - \frac{\alpha K_A}{\mu K_M}\varphi^0 - \frac{\beta K_B}{\mu K_M}\Phi^0 \\ \Phi^0 = -\frac{\varepsilon K_E}{K_{emgc}^2} + \frac{\alpha^2 K_A^2}{\mu K_M K_{emgc}^2} + \frac{\left(\frac{\xi K_S}{K_{emgc}^2} - \frac{\alpha\lambda K_A K_L}{\mu K_M K_{emgc}^2} \right)^2}{\frac{\eta K_F}{K_{emgc}^2} - \frac{\lambda^2 K_L^2}{\mu K_M K_{emgc}^2}} \\ \Psi^0 = -\frac{\xi K_S - \frac{\alpha\lambda K_A K_L}{\mu K_M}}{\eta K_F - \frac{\lambda^2 K_L^2}{\mu K_M}}\varphi^0 - \frac{\vartheta K_T - \frac{\beta\lambda K_B K_L}{\mu K_M}}{\eta K_F - \frac{\lambda^2 K_L^2}{\mu K_M}}\Phi^0 \end{array} \right) \quad (80)$$

The second equation in set (78) is responsible for the existence of the second eigenvectors. Their components can be composed as follows:

$$\begin{pmatrix} U^{0(1)} \\ \varphi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix} = \begin{pmatrix} U^{0(3)} \\ \varphi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix} = \begin{pmatrix} U^{0(5)} \\ \varphi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix} = \begin{pmatrix} U^{0(7)} \\ \varphi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} = \left(\begin{array}{c} U^0 = 0 \\ \varphi^0 = -\gamma + \frac{\beta^2}{\mu} + \frac{\left(\vartheta - \frac{\beta\lambda}{\mu}\right)^2}{\eta - \frac{\lambda^2}{\mu}} \\ \psi^0 = -\frac{\lambda}{\mu}\Psi^0 - \frac{\alpha}{\mu}\varphi^0 - \frac{\beta}{\mu}\Phi^0 \\ \Phi^0 = \zeta - \frac{\alpha\beta}{\mu} - \frac{\left(\xi - \frac{\alpha\lambda}{\mu}\right)\left(\vartheta - \frac{\beta\lambda}{\mu}\right)}{\eta - \frac{\lambda^2}{\mu}} \\ \Psi^0 = -\frac{\xi - \frac{\alpha\lambda}{\mu}}{\eta - \frac{\lambda^2}{\mu}}\varphi^0 - \frac{\vartheta - \frac{\beta\lambda}{\mu}}{\eta - \frac{\lambda^2}{\mu}}\Phi^0 \end{array} \right) \quad (81)$$

$$\begin{pmatrix} U^{0(9)} \\ \varphi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix} = \left(\begin{array}{c} U^0 = \left(e\varphi^0 + h\psi^0 + g\Phi^0 + f\Psi^0\right) / CK_{emgc}^2 \\ \varphi^0 = -\frac{\gamma K_G}{K_{emgc}^2} + \frac{\beta^2 K_B^2}{\mu K_M K_{emgc}^2} + \frac{\left(\frac{\vartheta K_T}{K_{emgc}^2} - \frac{\beta\lambda K_B K_L}{\mu K_M K_{emgc}^2} \right)^2}{\frac{\eta K_F}{K_{emgc}^2} - \frac{\lambda^2 K_L^2}{\mu K_M K_{emgc}^2}} \\ \psi^0 = \psi^0 = -\frac{\lambda K_L}{\mu K_M}\Psi^0 - \frac{\alpha K_A}{\mu K_M}\varphi^0 - \frac{\beta K_B}{\mu K_M}\Phi^0 \\ \Phi^0 = \frac{\zeta K_Z}{K_{emgc}^2} - \frac{\alpha\beta K_A K_B}{\mu K_M K_{emgc}^2} - \left(\frac{\xi K_S}{K_{emgc}^2} - \frac{\alpha\lambda K_A K_L}{\mu K_M K_{emgc}^2} \right) \frac{\vartheta K_T - \frac{\beta\lambda K_B K_L}{\mu K_M}}{\eta K_F - \frac{\lambda^2 K_L^2}{\mu K_M}} \\ \Psi^0 = -\frac{\xi K_S - \frac{\alpha\lambda K_A K_L}{\mu K_M}}{\eta K_F - \frac{\lambda^2 K_L^2}{\mu K_M}}\varphi^0 - \frac{\vartheta K_T - \frac{\beta\lambda K_B K_L}{\mu K_M}}{\eta K_F - \frac{\lambda^2 K_L^2}{\mu K_M}}\Phi^0 \end{array} \right) \quad (82)$$

The sixth case

Note that equations (47) can be also rewritten in the other possible forms that are different from the six case treated below. However, this order of equations is final for this research. The reader can also have some practice in mathematics to obtain the other possible forms if they exist. So, one can also regroup equations (47) as follows:

$$\begin{cases} \gamma\left(1 + mK_g^2/A\right)\Phi^0 + \vartheta\left(1 + mK_\vartheta^2/A\right)\Psi^0 \\ + \zeta\left(1 + mK_\zeta^2/A\right)\varphi^0 + \beta\left(1 + mK_\beta^2/A\right)\psi^0 = 0 \\ \vartheta\left(1 + mK_\vartheta^2/A\right)\Phi^0 + \eta\left(1 + mK_f^2/A\right)\Psi^0 \\ + \xi\left(1 + mK_\xi^2/A\right)\varphi^0 + \lambda\left(1 + mK_\lambda^2/A\right)\psi^0 = 0 \\ \zeta\left(1 + mK_\zeta^2/A\right)\Phi^0 + \xi\left(1 + mK_\xi^2/A\right)\Psi^0 \\ + \varepsilon\left(1 + mK_e^2/A\right)\varphi^0 + \alpha\left(1 + mK_\alpha^2/A\right)\psi^0 = 0 \\ \beta\left(1 + mK_\beta^2/A\right)\Phi^0 + \lambda\left(1 + mK_\lambda^2/A\right)\Psi^0 \\ + \alpha\left(1 + mK_\alpha^2/A\right)\varphi^0 + \mu\left(1 + mK_m^2/A\right)\psi^0 = 0 \end{cases} \quad (83)$$

The first equation in set (83) gives the following definition:

$$\Phi^0 = -\frac{\vartheta\left(A + mK_\vartheta^2\right)}{\gamma\left(A + mK_g^2\right)}\Psi^0 - \frac{\zeta\left(A + mK_\zeta^2\right)}{\gamma\left(A + mK_g^2\right)}\varphi^0 - \frac{\beta\left(A + mK_\beta^2\right)}{\gamma\left(A + mK_g^2\right)}\psi^0 \quad (84)$$

Definition (84) is then used in set (83) to reduce this set of four equations and afterward to deal with the following set of three homogeneous equations:

$$\begin{cases} \left(\frac{\eta\left(A + mK_f^2\right)}{A} - \frac{\vartheta^2\left(A + mK_\vartheta^2\right)^2}{A\gamma\left(A + mK_g^2\right)} \right)\Psi^0 + \left(\frac{\xi\left(A + mK_\xi^2\right)}{A} - \frac{\vartheta\zeta\left(A + mK_\zeta^2\right)\left(A + mK_\vartheta^2\right)}{A\gamma\left(A + mK_g^2\right)} \right)\varphi^0 \\ + \left(\frac{\lambda\left(A + mK_\lambda^2\right)}{A} - \frac{\beta\vartheta\left(A + mK_\beta^2\right)\left(A + mK_\vartheta^2\right)}{A\gamma\left(A + mK_g^2\right)} \right)\psi^0 = 0 \\ \left(\frac{\xi\left(A + mK_\xi^2\right)}{A} - \frac{\vartheta\zeta\left(A + mK_\zeta^2\right)\left(A + mK_\vartheta^2\right)}{A\gamma\left(A + mK_g^2\right)} \right)\Psi^0 + \left(\frac{\varepsilon\left(A + mK_e^2\right)}{A} - \frac{\zeta^2\left(A + mK_\zeta^2\right)^2}{A\gamma\left(A + mK_g^2\right)} \right)\varphi^0 \\ + \left(\frac{\alpha\left(A + mK_\alpha^2\right)}{A} - \frac{\beta\zeta\left(A + mK_\beta^2\right)\left(A + mK_\zeta^2\right)}{A\gamma\left(A + mK_g^2\right)} \right)\psi^0 = 0 \\ \left(\frac{\lambda\left(A + mK_\lambda^2\right)}{A} - \frac{\beta\vartheta\left(A + mK_\beta^2\right)\left(A + mK_\vartheta^2\right)}{A\gamma\left(A + mK_g^2\right)} \right)\Psi^0 + \left(\frac{\alpha\left(A + mK_\alpha^2\right)}{A} - \frac{\beta\zeta\left(A + mK_\beta^2\right)\left(A + mK_\zeta^2\right)}{A\gamma\left(A + mK_g^2\right)} \right)\varphi^0 \\ + \left(\frac{\mu\left(A + mK_m^2\right)}{A} - \frac{\beta^2\left(A + mK_\beta^2\right)^2}{A\gamma\left(A + mK_g^2\right)} \right)\psi^0 = 0 \end{cases} \quad (85)$$

It is convenient to exploit the first equation in set (85) for determination of the eigenvector component Ψ^0. It is defined by

$$\Psi^0 = -\frac{\xi\left(A + mK_\xi^2\right) - \frac{\vartheta\zeta\left(A + mK_\zeta^2\right)\left(A + mK_\vartheta^2\right)}{\gamma\left(A + mK_g^2\right)}}{\eta\left(A + mK_f^2\right) - \frac{\vartheta^2\left(A + mK_\vartheta^2\right)^2}{\gamma\left(A + mK_g^2\right)}}\varphi^0 - \frac{\lambda\left(A + mK_\lambda^2\right) - \frac{\beta\vartheta\left(A + mK_\beta^2\right)\left(A + mK_\vartheta^2\right)}{\gamma\left(A + mK_g^2\right)}}{\eta\left(A + mK_f^2\right) - \frac{\vartheta^2\left(A + mK_\vartheta^2\right)^2}{\gamma\left(A + mK_g^2\right)}}\psi^0 \quad (86)$$

In the final accord, definition (86) is used for reduction of equations' set (85). The reduced set of equations represents two homogeneous equations that can be expressed in the following form:

$$\begin{cases} 0=\left(\dfrac{\varepsilon\left(A+mK_e^2\right)}{A}-\dfrac{\zeta^2\left(A+mK_\zeta^2\right)^2}{A\gamma\left(A+mK_g^2\right)}-\dfrac{\left(\dfrac{\xi\left(A+mK_\xi^2\right)}{A}-\dfrac{\vartheta\zeta\left(A+mK_\zeta^2\right)\left(A+mK_\vartheta^2\right)}{A\gamma\left(A+mK_g^2\right)}\right)^2}{\dfrac{\eta\left(A+mK_f^2\right)}{A}-\dfrac{\vartheta^2\left(A+mK_\vartheta^2\right)^2}{A\gamma\left(A+mK_g^2\right)}}\right)\varphi^0 \\ +\left(\dfrac{\alpha\left(A+mK_\alpha^2\right)}{A}-\dfrac{\beta\zeta\left(A+mK_\beta^2\right)\left(A+mK_\zeta^2\right)}{A\gamma\left(A+mK_g^2\right)}\right)\psi^0 \\ -\left(\dfrac{\xi\left(A+mK_\xi^2\right)}{A}-\dfrac{\vartheta\zeta\left(A+mK_\zeta^2\right)\left(A+mK_\vartheta^2\right)}{A\gamma\left(A+mK_g^2\right)}\right)\dfrac{\lambda\left(A+mK_\lambda^2\right)-\dfrac{\beta\vartheta\left(A+mK_\beta^2\right)\left(A+mK_\vartheta^2\right)}{\gamma\left(A+mK_g^2\right)}}{\eta\left(A+mK_f^2\right)-\dfrac{\vartheta^2\left(A+mK_\vartheta^2\right)^2}{\gamma\left(A+mK_g^2\right)}}\psi^0 \\ 0=\left(\dfrac{\alpha\left(A+mK_\alpha^2\right)}{A}-\dfrac{\beta\zeta\left(A+mK_\beta^2\right)\left(A+mK_\zeta^2\right)}{A\gamma\left(A+mK_g^2\right)}\right)\varphi^0 \\ -\left(\dfrac{\xi\left(A+mK_\xi^2\right)}{A}-\dfrac{\vartheta\zeta\left(A+mK_\zeta^2\right)\left(A+mK_\vartheta^2\right)}{A\gamma\left(A+mK_g^2\right)}\right)\dfrac{\lambda\left(A+mK_\lambda^2\right)-\dfrac{\beta\vartheta\left(A+mK_\beta^2\right)\left(A+mK_\vartheta^2\right)}{\gamma\left(A+mK_g^2\right)}}{\eta\left(A+mK_f^2\right)-\dfrac{\vartheta^2\left(A+mK_\vartheta^2\right)^2}{\gamma\left(A+mK_g^2\right)}}\varphi^0 \\ +\left(\dfrac{\mu\left(A+mK_m^2\right)}{A}-\dfrac{\beta^2\left(A+mK_\beta^2\right)^2}{A\gamma\left(A+mK_g^2\right)}-\dfrac{\left(\dfrac{\lambda\left(A+mK_\lambda^2\right)}{A}-\dfrac{\beta\vartheta\left(A+mK_\beta^2\right)\left(A+mK_\vartheta^2\right)}{A\gamma\left(A+mK_g^2\right)}\right)^2}{\dfrac{\eta\left(A+mK_f^2\right)}{A}-\dfrac{\vartheta^2\left(A+mK_\vartheta^2\right)^2}{A\gamma\left(A+mK_g^2\right)}}\right)\psi^0 \end{cases} \tag{87}$$

Final equations' set (87) allows one to obtain all possible eigenvectors. With the first equation in the set, one can find that the first eigenvectors can be exposed in the following explicit forms:

$$\begin{pmatrix} U^{0(1)} \\ \varphi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix}=\begin{pmatrix} U^{0(3)} \\ \varphi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix}=\begin{pmatrix} U^{0(5)} \\ \varphi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix}=\begin{pmatrix} U^{0(7)} \\ \varphi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix} =\begin{pmatrix} U^0=0 \\ \varphi^0=\alpha-\dfrac{\beta\zeta}{\gamma}-\dfrac{\left(\xi-\dfrac{\vartheta\zeta}{\gamma}\right)\left(\lambda-\dfrac{\beta\vartheta}{\gamma}\right)}{\eta-\dfrac{\vartheta^2}{\gamma}} \\ \psi^0=-\varepsilon+\dfrac{\zeta^2}{\gamma}+\dfrac{\left(\xi-\dfrac{\vartheta\zeta}{\gamma}\right)^2}{\eta-\dfrac{\vartheta^2}{\gamma}} \\ \Phi^0=-\dfrac{\vartheta}{\gamma}\Psi^0-\dfrac{\zeta}{\gamma}\varphi^0-\dfrac{\beta}{\gamma}\psi^0 \\ \Psi^0=-\dfrac{\xi-\dfrac{\vartheta\zeta}{\gamma}}{\eta-\dfrac{\vartheta^2}{\gamma}}\varphi^0-\dfrac{\lambda-\dfrac{\beta\vartheta}{\gamma}}{\eta-\dfrac{\vartheta^2}{\gamma}}\psi^0 \end{pmatrix} \tag{88}$$

$$\begin{pmatrix} U^{0(9)} \\ \varphi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix}=\begin{pmatrix} U^0=\left(e\varphi^0+h\psi^0+g\Phi^0+f\Psi^0\right)/CK_{emgc}^2 \\ \varphi^0=\dfrac{\alpha K_A}{K_{emgc}^2}-\dfrac{\beta\zeta K_BK_Z}{\gamma K_GK_{emgc}^2}-\left(\dfrac{\xi K_S}{K_{emgc}^2}-\dfrac{\vartheta\zeta K_TK_Z}{\gamma K_GK_{emgc}^2}\right)\dfrac{\lambda K_L-\dfrac{\beta\vartheta K_BK_T}{\gamma K_G}}{\eta K_F-\dfrac{\vartheta^2K_T^2}{\gamma K_G}} \\ \psi^0=-\dfrac{\varepsilon K_E}{K_{emgc}^2}+\dfrac{\zeta^2K_Z^2}{\gamma K_GK_{emgc}^2}+\dfrac{\left(\dfrac{\xi K_S}{K_{emgc}^2}-\dfrac{\vartheta\zeta K_TK_Z}{\gamma K_GK_{emgc}^2}\right)^2}{\dfrac{\eta K_F}{K_{emgc}^2}-\dfrac{\vartheta^2K_T^2}{\gamma K_GK_{emgc}^2}} \\ \Phi^0=-\dfrac{\vartheta K_T}{\gamma K_G}\Psi^0-\dfrac{\zeta K_Z}{\gamma K_G}\varphi^0-\dfrac{\beta K_B}{\gamma K_G}\psi^0 \\ \Psi^0=-\dfrac{\xi K_S-\dfrac{\vartheta\zeta K_TK_Z}{\gamma K_G}}{\eta K_F-\dfrac{\vartheta^2K_T^2}{\gamma K_G}}\varphi^0-\dfrac{\lambda K_L-\dfrac{\beta\vartheta K_BK_T}{\gamma K_G}}{\eta K_F-\dfrac{\vartheta^2K_T^2}{\gamma K_G}}\psi^0 \end{pmatrix} \tag{89}$$

With the second equation in set (87), one can obtain the second eigenvectors. Their explicit forms can be demonstrated as follows:

$$\begin{pmatrix} U^{0(1)} \\ \varphi^{0(1)} \\ \psi^{0(1)} \\ \Phi^{0(1)} \\ \Psi^{0(1)} \end{pmatrix}=\begin{pmatrix} U^{0(3)} \\ \varphi^{0(3)} \\ \psi^{0(3)} \\ \Phi^{0(3)} \\ \Psi^{0(3)} \end{pmatrix}=\begin{pmatrix} U^{0(5)} \\ \varphi^{0(5)} \\ \psi^{0(5)} \\ \Phi^{0(5)} \\ \Psi^{0(5)} \end{pmatrix}=\begin{pmatrix} U^{0(7)} \\ \varphi^{0(7)} \\ \psi^{0(7)} \\ \Phi^{0(7)} \\ \Psi^{0(7)} \end{pmatrix}=\begin{pmatrix} U^0=0 \\ \varphi^0=-\mu+\dfrac{\beta^2}{\gamma}+\dfrac{\left(\lambda-\dfrac{\beta\vartheta}{\gamma}\right)^2}{\eta-\dfrac{\vartheta^2}{\gamma}} \\ \psi^0=\alpha-\dfrac{\beta\zeta}{\gamma}-\dfrac{\left(\xi-\dfrac{\vartheta\zeta}{\gamma}\right)\left(\lambda-\dfrac{\beta\vartheta}{\gamma}\right)}{\eta-\dfrac{\vartheta^2}{\gamma}} \\ \Phi^0=-\dfrac{\vartheta}{\gamma}\Psi^0-\dfrac{\zeta}{\gamma}\varphi^0-\dfrac{\beta}{\gamma}\psi^0 \\ \Psi^0=-\dfrac{\xi-\dfrac{\vartheta\zeta}{\gamma}}{\eta-\dfrac{\vartheta^2}{\gamma}}\varphi^0-\dfrac{\lambda-\dfrac{\beta\vartheta}{\gamma}}{\eta-\dfrac{\vartheta^2}{\gamma}}\psi^0 \end{pmatrix} \tag{90}$$

$$\begin{pmatrix} U^{0(9)} \\ \varphi^{0(9)} \\ \psi^{0(9)} \\ \Phi^{0(9)} \\ \Psi^{0(9)} \end{pmatrix}=\begin{pmatrix} U^0=\left(e\varphi^0+h\psi^0+g\Phi^0+f\Psi^0\right)/CK_{emgc}^2 \\ \varphi^0=-\dfrac{\mu K_M}{K_{emgc}^2}+\dfrac{\beta^2K_B^2}{\gamma K_GK_{emgc}^2}+\dfrac{\left(\dfrac{\lambda K_L}{K_{emgc}^2}-\dfrac{\beta\vartheta K_BK_T}{\gamma K_GK_{emgc}^2}\right)^2}{\dfrac{\eta K_F}{K_{emgc}^2}-\dfrac{\vartheta^2K_T^2}{\gamma K_GK_{emgc}^2}} \\ \psi^0=\dfrac{\alpha K_A}{K_{emgc}^2}-\dfrac{\beta\zeta K_BK_Z}{\gamma K_GK_{emgc}^2}-\left(\dfrac{\xi K_S}{K_{emgc}^2}-\dfrac{\vartheta\zeta K_TK_Z}{\gamma K_GK_{emgc}^2}\right)\dfrac{\lambda K_L-\dfrac{\beta\vartheta K_BK_T}{\gamma K_G}}{\eta K_F-\dfrac{\vartheta^2K_T^2}{\gamma K_G}} \\ \Phi^0=-\dfrac{\vartheta K_T}{\gamma K_G}\Psi^0-\dfrac{\zeta K_Z}{\gamma K_G}\varphi^0-\dfrac{\beta K_B}{\gamma K_G}\psi^0 \\ \Psi^0=-\dfrac{\xi K_S-\dfrac{\vartheta\zeta K_TK_Z}{\gamma K_G}}{\eta K_F-\dfrac{\vartheta^2K_T^2}{\gamma K_G}}\varphi^0-\dfrac{\lambda K_L-\dfrac{\beta\vartheta K_BK_T}{\gamma K_G}}{\eta K_F-\dfrac{\vartheta^2K_T^2}{\gamma K_G}}\psi^0 \end{pmatrix} \tag{91}$$

The reader can find that all the obtained eigenvector components do not depend on the phase velocity. This is true for all the treated six cases. This peculiarity can be further used for finding the propagation velocity of the acoustic wave when different boundary conditions will be applied.

CONCLUSION

This analysis has demonstrated that many possible eigenvectors can be revealed for the problem of the shear-horizontal acoustic wave propagation coupled with the electrical, magnetic, gravitational, and cogravitational potentials. This can be explained by the fact that in this case, any found apt eigenvector does not depend on the phase velocity. This peculiarity exists in certain directions of the transversely isotropic (6 *mm*) continua. Exploitation of each found eigenvector can give an unique solution for the propagation velocity of the acoustic wave. The existence of some found unique acoustic waves can dramatically depend on one of the extremely weak exchange effects: the magnetoelectric, gravitocogravitic, gravitoelectric, cogravitoelectric, gravitomagnetic, cogravitomagnetic effects. This possibility must be analytically demonstrated in the future, using the found eigenvectors. The obtained analytical results can be readily used for finding the propagation velocities of the acoustic waves managed by the solid surface, common interface between two solids, waveguide consisting of this film (plate), and more complicated configurations. Therefore, the obtained results can stimulate constitution of suitable technical devices based on some gravitation phenomena by experimentalists and engineers working with the transmitting, detecting, and converting of the electromagnetic waves' signals.

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